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ON COMMUTATIVITY OF ONE SIDED s-UNITAL RINGS WITH SOME POLYNOMIAL CONSTRAINTS

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In the present paper it is proved that a ring with unity 1 is commutative if and only if it satisfies either of the properties $[x^m y - x^n f(x^m y) x^p, x] = 0$ or $[yx^m - x^n f(x^m y) x^p, x] = 0$; where $m \geq 0$, $n \geq 0$, $p \geq 0$ are fixed non-negative integers and $f(t)$ is a polynomial in $t^2 \mathbf{Z}[t]$ varying with the pair of ring elements x, y. Further, the result has been extended to the one sided s-unital rings. Finally, in the underlying conditions the integral exponents are also allowed to vary with the choice of x, y and ring satisfies the Chacron's condition.

1. INTRODUCTION

Throughout, R will represent an associative ring. A ring R is called left (resp. right) s-unital if for each x in R, $x \in Rx$ (resp. $x \in xR$); R is called s-unital if R is both left and right s-unital. As usual $\mathbf{Z}[t]$ is the totality of polynomials in t with coefficients in $\mathbf{Z}$, the ring of integers and for any x, y in R the symbol $[x, y]$ stands for the commutator $xy - yx$. For fixed non-negative integers $m \geq 0$, $n \geq 0$, $p \geq 0$, consider the following ring properties :

(*) For each x, y in R there exists $f(t)$ in $t^2 \mathbf{Z}[t]$ such that $[x^m y - x^n f(x^m y) x^p, x] = 0$.

(*)$'$ For each x, y in R there exists $f(t)$ in $t^2 \mathbf{Z}[t]$ such that $[y x^m - x^n f(x^m y) x^p, x] = 0$.

In an attempt to generalize famous Jacobson's $x^n = x$ theorem it was proved by Herstein[3] that if for each x, y in R there exists a polynomial $f(t)$ in $t^2 \mathbf{Z}[t]$ such that $[x - f(x), y] = 0$, then R is commutative. Further, Putcha and Yaqub[9] established that if for each x, y in R there exists a polynomial $f(t) \in t^2 \mathbf{Z}[t]$ such that $xy - f(xy)$ is central, then R^2 is central. Recently Bell *et al.*[1] obtained the commutativity of R with unity 1 satisfying the polynomial identity $[xy - f(xy), x] = 0$, where $f(t) \in t^2 \mathbf{Z}[t]$. In the present paper our objective is to further extend the above results to the rings satisfying conditions of the form (*) or (*)$'$, where $f(t)$ is no longer fixed, rather it is varying with x and y.

2. Commutativity of Rings with Unity

Theorem 1 — If R is a ring with unity 1 satisfying either of the properties (*) or (*)′, then R is commutative (and conversely).

In preparation for the proof of the above theorem, we begin with the following types of rings :

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

$(\text{i})_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

$(\text{i})_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(ii) $M_\sigma(k) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \Big/ a, b \in K \right\}$, where K is a finite field with a non-trivial automorphism σ.

(iii) A non-commutative division ring.

(iv) $S = \langle 1 \rangle + T$, T a non-commutative radical subring of S.

(v) $S = \langle 1 \rangle + T$, T a non-commutative subring of S such that $T[T, T] = [T, T]T = 0$.

Recently, Streb[13] classified non-commutative rings, which has been used effectively to obtain a number of commutativity theorems (cf. Komatsu and coworkers[7, 8]). It can be observed from the proof of Corollary 1 of Streb[13] that if R is a non-commutative ring with unity 1, then there exists a factor subring of R, which is of type (i), (ii), (iii), (iv) or (v). This result gives the following lemma, which plays an important role in the subsequent study (cf. Komatsu and Tominaga[8], Lemma 1).

Lemma 1 — Let C be a ring property which is inherited by factor subrings. If no rings of type (i), (ii), (iii), (iv) or (v) satisfy C, then every ring with unity 1 satisfying C is commutative.

The proof of the following lemmas can be seen in Herstien[3] and Komatsu *et al.*[7] respectively .

Lemma 2 — Let R be a ring in which for every x, y in R there exists a polynomial $f(t) \in t^2\mathbb{Z}[t]$ such that $[x - f(x), y] = 0$. Then R is commutative.

Lemma 3 — Let R be a left s-unital not a right s-unital, then R has a factor subring of type $(\text{i})_l$.

It is to remark that each time we shall prove the results for rings satisfying the property (*). The proofs for (*)′ are similar to those of (*).

Suppose that R is a ring with unity 1 satisfying (*). If x is a unit in R and $y \in R$, then there exists a polynomial $f(t) \in t^2\mathbb{Z}[t]$ such that

$$[y - x^n f(y) x^p, x] = [x^m x^{-m} y - x^n f(x^m x^{-m} y) x^p, x] = 0.$$

Now using Lemma 2(1) of Komatsu and Tominaga[8] for the case $m = 0$, we get the following :

Lemma 4 — Let R be a ring with unity 1 satisfying (*). Then for each unit $x \in R$ and each $y \in R$ there exists a polynomial $h(t) \in t^2\mathbf{Z}[t]$ such that $[x, y - h(y)] = 0$.

PROOF OF THEOREM 1 — First consider the ring of type (i). Then in $M_2\,GF(p)$, p a prime we find that $[e_{11}^m e_{12} - e_{11}^n f(e_{11}^m e_{12}) e_{11}^p, e_{11}] = -e_{12} \neq 0$ for every $f(t)$ in $t^2\mathbf{Z}[t]$. Thus no rings of type (i) satisfy (*).

Next, consider the ring $M_\sigma(K)$ a ring of type (ii). Now choose

$$x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} (\sigma(a) \neq a),\ y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

such that $[x^m y - x^n f(x^m y) x^p, x]$

$$= \left[\begin{pmatrix} 0 & a^m \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}\right] = a^m(\sigma(a) - a)\,e_{12} \neq 0$$

for every $f(t) \in t^2\mathrm{Z}[t]$.

Further, let R be a ring of type (iii). If x is a unit in R and $y \in R$, then in view of Lemma 4, there exists a polynomial $f(t) \in t^2\mathbf{Z}[t]$ such that $[x, y - f(y)] = 0$. Thus by Lemma 2, R is commutative, a contradiction.

Now let R be a ring of type (iv). Suppse that $a, b \in T$. Since $1 - a$ is a unit, in view of Lemma 4 we find that there exists a polynomial $f(t) \in t^2\mathbf{Z}[t]$ such that $[b - f(b), a] = -[b - f(b), 1 - a] = 0$. Hence, by Lemma 2, T is commutative. This is impossible.

Finally, suppose that R is of type (v). Then for each $a, b \in T$ there exists $f(t) \in t^2\mathbf{Z}[t]$ such that

$$[a, b] = (1+a)^m[a, b] = (1+a)^n[a, f((1+a)^m b)](1+a)^p = 0.$$

This is a contradiction.

Hence, this proves that no rings of type (i), (ii), (iii), (iv) or (v) satisfy (*) and by Lemma 1, R is commutative.

Corollary 1 — Let $m \geq 0$, $n \geq 0$, $p \geq 0$ be fixed integers and R be a ring with unity 1. If for each x, y in R there exists an integer $s = s(x, y) > 1$ such that either $[x^m y - x^n (x^m y)^s x^p, x] = 0$ or $[y x^m - x^n (x^m y)^s x^p, x] = 0$, then R is commutative (and conversely).

3. COMMUTATIVITY OF ONE SIDED s-UNITAL RINGS

The existence of non-commutative ring R with R^2 being central rules out the possible generalization of the above theorem for arbitrary rings. However, if R is a left (resp. right) s-unital ring satisfying (*) (resp. (*)'), then first paragraph of the proof of the above theorem also shows that no rings of type $(\mathrm{i})_l$ (resp. $(\mathrm{i})_r$) satisfy (*) (resp. (*)') and by Lemma 3 (resp. dual of Lemma 3), R is right (resp. left) s-unital. Hence in both cases R is s-unital and in view of Proposition 1 of Hirano *et al.*[4], we may assume that R has unity 1. Thus Theorem 1, yields the following :

Theorem 2 — Let R be a left (resp. right) s-unital ring satisfying (*) (resp. (*)′). Then R is commutative (and conversely).

The following example demonstrates that there are non-commutative left (resp. right) s-unital rings satisfying (*)′ (resp. (*)).

Example — Let

$$R_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

$$\left(\text{resp. } R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \right)$$

be subring of 2×2 matrices over $GF(2)$. Then for any fixed positive integers m, n, p and polynomial $f(t) \in t^2 \mathbf{Z}[t]$, R_1 (resp. R_2) satisfies (*)′ (resp. (*)). However, R_1 (resp. R_2) is a non-commutative left (resp. right) s-unital ring.

As a corollary to the above theorem, we get the following result improving earlier results (see Quadri *et al.*[10], Theorem; Quadri and Ashraf[11], Theorem A and Searcoid and MacHale[12], Theorem).

Corollary 2 — Let R be a left (resp. right) s-unital ring in which for every x, y in R there exists an integer $n = n(x, y) > 1$ such that $[xy - (xy)^n, x] = 0$ (resp. $[yx - (xy)^n, x] = 0$). Then R is commutative (and conversely).

4. Extensions to Variable Exponents

If the integral exponents m, n and p in the conditions (*) and (*)′ are also allowed to vary with the pair of ring elements x, y, then following are the weaker version of conditions (*) and (*)′ respectively.

(**) For each x, y in R there exist integers $m \geq 0$, $n \geq 0$, $p \geq 0$, and polynomial $f(t)$ in $t^2 \mathbf{Z}[t]$ such that $[x^m y - x^n f(x^m y) x^p, x] = 0$.

(**)′ For each x, y in R there exist integers $m \geq 0$, $n \geq 0$, $p \geq 0$ and polynomial $f(t)$ in $t^2 \mathbf{Z}[t]$ such that $[yx^m - x^n f(x^m y) x^p, x] = 0$.

The following condition has been introduced by Chacron.

(CH) For each x, y in R there exist $f(t)$, $g(t)$ in $t^2 \mathbf{Z}[t]$ such that

$[x - f(x), y - f(y)] = 0$.

Very recently Komatsu and Tominaga proved that if R is a noncommutative ring with unity 1 satisfying (CH), then there exists a factor subring of R which is of type (i) or (ii) (cf. Komatsu and Tominaga[6], Corollary 1]). Thus in view of the proofs of Theorem 1 and Theorem 2, we can prove the following :

Theorem 3 — Let R be a ring with unity 1 satisfying either of the properties (**) or (**)′. Suppose further that R satisfies (CH), then R is commutative (and conversely),

Theorem 4 — Let R be a left (resp. right) s-unital ring satisfying (**) (resp. (**)′). Suppose further that R satisfies (CH), then R is commutative (and conversely).

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