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ON NONLINEAR VARIATIONAL INEQUALITIES

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In this paper, we develop the existence theory of nonlinear variational inequality problem in the setting of reflexive Banach spaces and Hausdorff topological vector spaces separately.

1. INTRODUCTION

Variational inequalities have not only stimulated new and deep results in many different branches of mathematics and engineering sciences but also provide us a unified and general framework for studying many unilateral free and moving boundary value problems arising in contact problems in elastostatics, fluid flow through porous media etc. Recently various extensions and generalizations of variational inequalities have been made and analyzed. An important generalization of variational inequality problem is the following nonlinear variational inequality problem :

Let $\langle X^*, X \rangle$ be a dual system of locally convex spaces and let K be a closed convex nonempty set in X.

Problem 1 — Find $u \in K$ such that

$$\langle T(u), v - u \rangle + j(v) - j(u) \geq \langle A(u), v - u \rangle, \text{ for all } v \in K, \qquad \ldots (1)$$

where $T, A : K \to X^*$ are nonlinear mappings and $j : X \to R$ is convex lower semicontinuous, nondifferentiable and proper functional.

If $A(u) = f \in X^*$ (dual of reflexive Banach space X) then Problem 1 has been studied by Brezis[1], Cohen[3] and Glowinski[5] by using different techniques. The variational inequality (1) arises in elastoplasticity, we refer to Reddy[9] and references therein. Note that the existence theory of Problem 1 in the setting of Hilbert space H is quite similar as given by Glowinski[5] for the case $A(u) = f \in H^*$, therefore, we do not consider this problem in Hilbert spaces. In this paper we develop the existence theory of Problem 1 in the setting of reflexive Banach spaces and Hausdorff topological vector spaces separately.

We need the following Definition 1 and Theorem 1 due to Fan[4] (see also Chen[2] and Isac[6]) and Theorem 2 due to Lin[7] in order to prove Theorem 3 and Theorem 4 respectively.

Definition 1 — A multivalued mapping $F : X \to 2^X$ is called the KKM mapping if for every finite subset $\{u_1, u_2, ..., u_n\}$ of X, $\text{conv}\{u_1, u_2, ..., u_n\} \subset \bigcup_{i=1}^{n} F(u_i)$, where conv (A) denotes convex hull of A.

Theorem 1 — Let K be a nonempty set in a Hausdorff topological vector space X. Let the multivalued mapping $F : K \to 2^X$ be a KKM mapping. If $F(u)$ is closed for all $u \in K$ and is compact for atleast one $u \in K$ then $\bigcap_{u \in K} F(u) \neq \phi$.

Theorem 2 — Let K be a compact convex nonempty set in a Hausdorff topological vector space X. Let B be a subset of $K \times K$ having the following properties :

(i) for each $u \in K, (u, u) \in B$;

(ii) for each $u \in K$, the set $B_u = \{v \in K : (u, v) \in B\}$ is closed;

(iii) for each $v \in K$, the set $B_v = \{u \in K : (u, v) \notin B\}$ is convex.

Then there exists a point $v_0 \in K$ such that $K \times \{v_0\} \subset B$.

2. Problem 1 in the Setting of Reflexive Banach Spaces

Throughout the section we denote by X, a reflexive Banach space, X^* a dual of X and $\langle . , . \rangle$ a continuous pairing between X^* and X. We prove the following lemma, a generalization of a lemma due to Minty[8], which will be needed in the proof of Theorem 3.

Lemma 1 — Let K be a nonempty convex set in X. Let the mappings $T, A : K \to X^*$ be both hemicontinuous and be monotone and antimonotone respectively. Also let the functional j on K be a convex, lower semicontinuous, nondifferentiable and proper. Then the following two problems are equivalent :

(i) $u \in K$, $\langle T(u), v - u \rangle + j(v) - j(u) \geq \langle A(u), v - u \rangle$, for all $v \in K$;

(ii) $u \in K$, $\langle T(v), v - u \rangle + j(v) - j(u) \geq \langle A(v), v - u \rangle$, for all $v \in K$.

Proof : (i) $\Rightarrow$ (ii). This is directly followed from the definitions of monotonicity and antimonotonicity.

(ii) $\Rightarrow$ (i). Suppose that u satisfies (ii). For every $v \in K$, $0 < t < 1$, we have

$$\langle T(tv + (1 - t)u), tv + (1 - t)u - u \rangle + j(tv + (1 - t)u) - j(u)$$

$$\geq \langle A(tv + (1 - t)u), tv + (1 - t)u - u \rangle.$$

By using the convexity of j, we have

$$t \langle T(u - t(v - u)), v - u \rangle + tj(v) - tj(u) \geq t \langle A(u - t(v - u)), v - u \rangle.$$

After dividing the above inequality by t we take limit as t tends to 0^+ then we have, by using hemicontinuity of T and A,

$$\langle T(u),\ v-u\rangle + j(v) - j(u) \geq \langle A(u),\ v-u\ \rangle, \text{ for all } v\in K$$

and the proof is complete.

Now we are able to prove the main result of this section.

Theorem 3 — Let K be a bounded, closed, convex and nonempty set in X. Let the mappings $T, A : K \to X^*$ and the functional j on X be same as in Lemma 1. Then Problem 1 is solvable. Moreover, if T is strictly monotone and A is strictly antimonotone then the solution of Problem 1 is unique.

PROOF : *Existence* — Let the multivalued mappings $F_1, F_2 : K \to 2^K$ be defined as

$$F_1(v) = \{u\in K : \langle T(u), v-u\rangle + j(v) - j(u)$$
$$\geq \langle A(u),\ v-u\rangle\} \text{ for each } v\in K \text{ and}$$
$$F_2(v) = \{u\in K : \langle T(v), v-u\rangle + j(v) - j(u)$$
$$\geq \langle A(v),\ v-u\rangle\} \text{ for each } v\in K, \text{ respectively.}$$

We claim that F_1 is a KKM mapping on K. Indeed if we assume $\{u_1, u_2, ..., u_n\} \subset K$, $\sum_{i=1}^{n} t_i = 1$, $t_i > 0$ and $u = \sum_{i=1}^{n} t_i u_i \notin \sum_{i=1}^{n} F_1(u_i)$, then

$$\langle T(u),\ u_i - u\rangle + j(u_i) - j(u) < \langle A(u),\ u_i - u\rangle \text{ for all } u_i\in K$$

or
$$\langle T(u) - A(u), u_i\rangle + j(u_i) < \langle T(u) - A(u),\ u\rangle + j(u), \text{ for all } u_i\in K.$$

Thus,

$$\langle T(u) - A(u), u\rangle + j(u) = \langle T(u) - A(u),\ \sum_{i=1}^{n} t_i u_i\rangle + j\left(\sum_{i=1}^{n} t_i u_i\right)$$
$$\leq \sum_{i=1}^{n} t_i (\langle T(u) - A(u),\ u_i\rangle + j(u_i))$$
$$< \langle T(u) - A(u), u\rangle + j(u)$$

which is a contradiction, so we have

$$\text{conv}\{u_1, u_2, ..., u_n\} \subset \sum_{i=1}^{n} F_1(u_i).$$

Next we claim that $F_1(v) \subset F_2(v)$ for every $v\in K$. Indeed, let $u\in F_1(v)$ then we have, by using monotonicity of T and antimonotonicity of A,

$$\langle T(v), v-u\rangle + j(v) - j(u) \geq \langle A(v), v-u\rangle,$$

i.e. $u \in F_2(v)$. Since u was an arbitrary element of $F_1(v)$ so we have $F_1(v) \subset F_2(v)$ which implies that F_2 is a KKM mapping on K. Hence, by Lemma 1, we have $\bigcap_{v\in K} F_1(v) = \bigcap_{v\in K} F_2(v)$.

Since the pairing $\langle .,. \rangle$ is continuous and the functional j is lower semicontinuous therefore, for every $v \in K$, $F_2(v)$ is closed. It is also clear that K is weakly compact set in X with weak topology. Hence $F_2(v)$ is weakly compact set in K since $F_2(v) \subset K$ for each $v \in K$. By Theorem 1, we have

$$\bigcap_{v\in K} F_1(v) = \bigcap_{v\in K} F_2(v) \neq \phi.$$

Hence there exists an $u_0 \in K$ such that

$$\langle T(u_0), v-u_0\rangle + j(v) - j(u_0) \geq \langle A(u_0), v-u_0\rangle \text{ for all } v \in K.$$

Uniqueness — Let u_1 and u_2 be the solutions of Problem 1, i.e.,

$$\langle T(u_1), v-u_1\rangle + j(v) - j(u_1) \geq \langle A(u_1), v-u_1\rangle \text{ for all } v \in K. \quad \ldots (2)$$

and

$$\langle T(u_2), v-u_2\rangle + j(v) - j(u_2) \geq \langle A(u_2), v-u_2\rangle \text{ for all } v \in K. \quad \ldots (3)$$

Substituting $v = u_2$ in (2) and $v = u_1$ in (3) and then adding we obtain

$$\langle T(u_1) - T(u_2),\ u_1 - u_2\rangle \leq \langle A(u_1) - A(u_2), u_1 - u_2\rangle < 0,$$

since A is strictly antimonotone. Hence, by monotonicity of T, we have

$$\langle T(u_1) - T(u_2),\ u_1 - u_2\rangle = 0$$

which implies $u_1 = u_2$, since T is strictly monotone.

3. Problem 1 in the Setting of Hausdorff Topological Vector Spaces

Let $\langle X^*, X\rangle$ be a dual system of locally convex spaces and let the bilinear form $\langle .,. \rangle$ be continuous. If K is a closed convex nonempty set and X in D is a compact convex nonempty set in K then we prove the following main result of this section :

Theorem 4 — Let the mappings $T, A : K \to X^*$ be nonlinear continuous and let the functional j on K be as same in Lemma 1. If $\langle T(u), 0\rangle \geq \langle A(u), 0\rangle$ for every $u \in D$ then there exists a point $u_0 \in D$ such that u_0 is a solution of Problem 1.

Proof : Let

$$E = \{(u, v) \in D \times D : \langle T(v), u-v\rangle + j(u) - j(v) \geq \langle A(v), u-v\rangle\}.$$

It is clear that $(u, u) \in E$ for every $u \in D$. For each $u \in D$, the set $E_u = \{v \in D : \langle T(v), u - v \rangle + j(u) - j(v) \geq \langle A(v), u - v \rangle\}$ is closed as can be easily seen from the continuity of $\langle ., . \rangle$ and lower semicontinuity of j. Now we claim that for each $v \in D$, the set $E_v = \{u \in D : \langle T(v), u - v \rangle + j(u) - j(v) < \langle A(v), u - v \rangle$ is convex. Indeed, let $u_1, u_2 \in E_v$ and $t, s \geq 0$ such that $t + s = 1$ then we have

$$\langle T(v), tu_1 - tv \rangle + tj(u_1) - tj(v) < \langle A(v), tu_1 - tv \rangle$$

and

$$\langle T(v), su_2 - sv \rangle + sj(u_2) - sj(v) < \langle A(v), su_2 - sv \rangle.$$

After adding the above two inequalities we use convexity of j and $t + s = 1$ then we obtain

$$\langle T(v), (tu_1 + su_2) - v \rangle + j(tu_1 + su_2) - j(v) < \langle A(v), (tu_1 + su_2) - v \rangle.$$

Since D is convex, so we can write $u_3 = (tu_1 + su_2) \in D$. Hence we have $u_3 \in E_v$ for each $v \in D$. Therefore, by Theorem 2, there exists a point $u_0 \in D$ such that $K \times \{u_0\} \subset E$ which implies that $u_0 \in D$ such that $\langle T(u_0), u - u_0 \rangle + j(u_0) - j(u) \geq \langle A(u_0), u - u_0 \rangle$ for all $u \in D$.

The proof is complete.

REFERENCES

1. H. Brezis, *Ann. de l'Inst. Fourier* **18** (1968), 115-75.
2. G. Y. Chen, *JOTA* **76** (3) (1992), 445-56.
3. G. Cohen, *JOTA* **59**(2) (1988), 325-33.
4. K. Fan, *Math. Annal.* **266** (1984), 519-37.
5. R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.
6. G. Isac, *J. Fac. Sci. Univ. of Tokyo* **37** (1990), 109-27.
7. T. C. Lin, *Bull. Austral. Math. Soc.* **34** (1986), 107-17.
8. G. J. Minty, *Duke Math. Oper. Res.* **1** (1976), 260-66.
9. B. D. Reddy, *Nonlinear Analysis TMA* **19** (1992), 1071-89.