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MULTI-VALUED MAPPINGS AND CLEAVABILITY

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We define and study multi-valued cleavability of topological spaces and extend some results on single-valued cleavability to this case. For example we prove that if a space X is perfectly multi-valued cleavable over the class of metrizable spaces (σ-spaces, Moore spaces, Fréchet-Urysohn spaces), then X is also such a space.

1. INTRODUCTION, PRELIMINARIES

Different types of cleavability (originally named "splittability") of a topological space were introduced by Arhangel'skii[2].

In the last few years many papers[3, 5, 6, 8, 9, 12, 13, 16; 23-30] concerning cleavability were published especially the survey papers by Arhangel'skii[1] and Kočinac *et al.*[30].

Here we define and study various types of multi-valued cleavability by using multi-valued mappings instead of single-valued. Among other, we prove that if a space X is perfectly mv-cleavable over the class of metrizable spaces, then X is also metrizable.

Throughout this paper single-valued mappings will be denoted by f, g, h, ..., while the capital letters F, G, ... will denote multi-valued mappings (briefly, mv-mappings).

Our topological notation and terminology are standard as in Arhangel'skii and Ponomarev[7] and Engelking[17] for general concepts in Berge[14] and Fedorchuk and Filippov[18] for multi-valued mappings, and in Michael[31] for generalized metric spaces. By τ we always denote an infinite cardinal number. As in Hodel[21] we use w, nw, L, hL, c, s, e, χ, ψ and $\mathcal{K}$ to denote the following cardinal functions : the weight, netweight, Lindelöf number, hereditary Lindelöf number, cellularity, spread, extent, character, pseudocharacter and the number of compact subspaces. $iw(X)$ = min {τ :

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there exists a continuous bijection from X onto a space Y with $w(Y) \leq \tau\}$. $cL(X)$ denotes the smallest cardinal τ such that for any closed $A \subset X$ and any family $\mathcal{U}$ of open subsets of X for which $A \subset \bigcup\mathcal{U}$ there is a subfamily $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \leq \tau$ and $A \subset \overline{\bigcup\mathcal{V}}$ (see, for example, Kočinac *et al.*[30]). $\Psi(X)$ is the smallest cardinal τ such that every closed set in X is the intersection of $\leq \tau$ open sets; if $\Psi(X)$ is countable we say that X is a perfect space.

Let $F : X \to Y$ be a multi-valued mapping from X into Y. (Note that we suppose that $F(x)$ is closed in Y for every $x \in X$.) For $y \in Y$ we write $F^{-1}y = \{x \in X : y \in F(x)\}$. If A is a subset of X and B is a subset of Y we put (as usual)

$$F(A) = \bigcup \{F(x) : x \in A\} = \{y \in Y : F^{-1}(y) \cap A \neq \phi\},$$

$$F^{\#}(A) = \{y \in Y : F^{-1}(y) \subset A\},$$

$$F^{-}(B) = \{x \in X : F(x) \cap B \neq \phi\},$$

$$F^{+}(B) = \{x \in X : F(x) \subset B\}.$$

It is easy to check the following relations (for $A \subset X$, $B \subset Y$) :

$$F^{-} F^{\#} (A) \subset A \subset F^{+} F(A) \subset F^{-} F(A); \quad \dots (1)$$

$$FF^{+} (B) \subset B \subset F^{\#} \; F^{-} (B) \subset FF^{-} (B) \quad \dots (2)$$

$$X \setminus F^{+}(B) = F^{-}(Y \setminus B); \quad \dots (3)$$

$$Y \setminus F^{\#}(A) = F(X \setminus A); \quad \dots (4)$$

$$A_1 \cap A_2 = \phi \text{ iff } F(A_1) \cap F^{\#} (A_2) = \phi; \quad \dots (5)$$

$$B_1 \cap B_2 = \phi \text{ iff } F^{-} (B_1) \cap F^{+} (B_2) = \phi. \quad \dots (6)$$

A mapping $F : X \to Y$ is USC (resp. LSC) if for every closed (resp. open) set $B \subset Y$ the set $F^{-} (B)$ is closed (open) in X; equivalently, F is USC (LSC) iff for every open (closed) set $M \subset Y$ the set $F^{+}(M)$ is open (closed) in X. Recall also that F is USC iff for every $x \in X$ and every neighbourhood V of the set $F(x)$ there exists a neighbourhood U of x such that $F(U) \subset V$. If F is both USC and LSC we say that F is continuous (briefly, C).

Definition 1.1 — A multi-valued mapping $F : X \to Y$ is said to be 'closed' ('open') if for every closed (open) set $A \subset X$, the set $F(A)$ is closed (open) in Y. ■

It is known that $F : X \to Y$ is closed (open) iff F^- is USC (LSC), and that F is closed iff for every $y \in Y$ and every neighbourhood U of $F^{-1}(y)$ there exists a neighbourhood V of y such that $F^-(V) \subset U$ (Ponomarev[33]).

Definition 1.2[33] — A multi-valued mapping $F : X \to Y$ is said to be perfect if the following conditions hold : (1) F is USC and closed; (2) F is compact-valued, i.e., for every $x \in X$, $F(x)$ is compact in Y; (3) F^- is compact-valued, i.e. for every $y \in Y$, $F^{-1}(y)$ is compact in X. ■

In what follows the following result of Yu. M. Smirnov (for the proof see Ponomarev[33]) will be used.

Proposition 1.3 — $F : X \to Y$ is perfect if and only if there are a space Z (= the graph of F) and single-valued perfect mappings $g : Z \to X$ and $h : Z \to Y$ such that $F = hg^{-1}$. ■

Let us note that perfectness of F implies perfectness of F^- because of the relation $F^- = gh^{-1}$.

Recall also : (1) if X and Y are compact and $F : X \to Y$ is USC, then F is perfect[33]; (2) if X is compact and F is USC and compact-valued, then Y is compact (more general, if K is compact in X, then $F(K)$ is compact in Y) (see Fedorchuk and Filippov[18]).

2. Multi-valued Cleavability

We say that a multi-valued mapping $F : X \to Y$ 'cleaves X along a subset A' $\subset X$ if $F^-F(A) = A$ holds. (By (1), then $F^+F(A) = A$ also holds).

Remark 2.1 : If $F : X \to Y$ cleaves X along a set $A \subset X$ and if f is a single-valued selection of F, then f also cleaves X along A.

Remark 2.2 : According to (2) one can naturally consider cleavability of Y along a subset $B \subset Y$ when $B = FF^-(B)$ holds (and then $B = F^{\#}F^-(B)$). ■

Let $F : X \to Y$ be an mv-mapping and let C_F denote the family of all subsets A of X such that F cleaves X along A : $C_F = \{A \subset X : F^-F(A) = A\}$.

Proposition 2.3 — The family C_F is a complemented lattice.

Proof : Let $A \in C_F$. Then $F(A) \cap F(X \setminus A) = \phi$ implies, by (6), $\phi = F^-F(X\setminus A) \cap F^+F(A) = F^-F(X\setminus A) \cap F^-F(A) = F^-F(X\setminus A) \cap A$ and so $F^-F(X\setminus A) = X \setminus A$, i.e. $X \setminus A \in C_F$. Let $A, B \in C_F$. Then :

(i) $F^-F(A \cup B) = F^-F(A) \cup F^-F(B) = A \cup B$, i.e. $A \cup B \in C_F$;

(ii) $X \setminus (A \cap B) = (X\setminus A) \cup (X\setminus B) \in C_F$, so that $A \cap B \in C_F$. ■

In a similar way one can check that $F(C_F)$ is also a complemented lattice.

Now we give a general definition of mv-cleavability.

Definition 2.4 — Let $\mathcal{F}$ be a class of multi-valued mappings, $\mathcal{P}$ a class of topological spaces. A space X is said to be '$\mathcal{F}$-mv-cleavable over $\mathcal{P}$' (resp.

'$\mathcal{F}$-pointwise mv-cleavable over $\mathcal{P}$') iff for every $A \subset X$ (resp. every $x \in X$) there exist a space $Y \in \mathcal{P}$ and a mapping $F \in \mathcal{F}$, $F : X \to Y$, such that $Y = F(X)$ and $F^{-}F(A) = A$ (resp. $F^{-}F(X) = \{x\}$). If $\mathcal{P}$ is the family of all subsets of a space Y we say that X is '$\mathcal{F}$-mv-cleavable over Y'. ■

When $\mathcal{F}$ is the class of all USC, LSC, C, closed, open, perfect, ... mappings we shall use the corresponding term instead of $\mathcal{F}$ in this definition (for example, USC-mv-cleavability over $\mathcal{P}$, and so on).

Recall that a mapping $F : X \to Y$ is called 'injective' if $x_1, x_2 \in X, x_1 \neq x_2$ implies $F(x_1) \bigcap F(x_2) = \phi$. Let us note that if there exists a multi-valued injection $F \in \mathcal{F}$ from X onto a space $Y \in \mathcal{P}$, then, obviously, X is $\mathcal{F}$-mv-cleavable over $\mathcal{P}$. So, $\mathcal{F}$-mv-cleavability may be viewed as a relative version of the concept of a multi-valued injection (from $\mathcal{F}$).

The following three natural general questions concerning mv-cleavability are the most important and they will be considered below :

Question A : Which spaces X are $\mathcal{F}$-mv-cleavable over a class $\mathcal{P}$ (along a collection of subsets of X) ?

Question B : If a space X is $\mathcal{F}$-mv-cleavable over $\mathcal{P}$, does X belong to $\mathcal{P}$?

Question C : When does $\mathcal{F}$-mv-cleavability of X over $\mathcal{P}$ imply the existence of an injection $F \in \mathcal{F}$ from X onto some $Y \in \mathcal{P}$?

Let us point out the following result.

Theorem 2.5 — If $f : X \to Y$ is a closed and open single-valued mapping and X is C-mv-cleavable over a space Z (over a class $\mathcal{P}$), then Y is also C-mv-cleavable over Z (over $\mathcal{P}$).

PROOF : The mapping $F : Y \to X$ defined by $F(y) = f^{-1}(y)$, $y \in Y$, is LSC (because f is open) and USC (because f is closed). Let $B \subset Y$. Take a continuous mv-mapping $G : X \to Z$ such that $G^{-}GF(B) = F(B)$. Let $H = GF$. Then H is USC and LSC and, on the other hand, $H^{-}H(B) = (F^{-}G^{-})\,GF(B) = F^{-}F(B) = ff^{-1}(B) = B$. This means that Y is C-mv-cleavable over Z. ■

Remark 2.6 : For single-valued cleavability the previous result need not be true. Even open perfect mappings do not preserve cleavability over a space. Keldysh[22] has shown that for any $n > 3$ there exists an open perfect mapping from I^3 onto I^n. But it is known that I^4, for example, is not cleavable over I^3.

In what follows we shall often use the following lemma.

Lemma 2.7 — Let a perfect mv-mapping $F : X \to Y$ cleaves X along a subset $A \subset X$ and let $\mathcal{P}$ be a hereditary topological property which is both an invariant and inverse invariant of perfect single-valued mappings. Then :

(1) If Z, g, h are as in Proposition 1.3, then h cleaves Z along the set $g^{-1}(A)$;

(2) A has $\mathcal{P}$.

In particular, if X is perfectly mv-cleavable over $\mathcal{P}$, then X is hereditarily $\mathcal{P}$.

PROOF : (1) We have $A = F^{-}F(A) = g(h^{-1}hg^{-1}(A))$. If one assumes that (1) is not true, then form $z \in h^{-1}hg^{-1}(A) \setminus g^{-1}(A)$ it follows $g(z) \notin A$ which is impossible because of $g(z) \in gh^{-1}hg^{-1}(A) = A$.

(2) Since $h^{-1}hg^{-1}(A) = g^{-1}(A)$ the mapping $h \mid g^{-1}(A) : g^{-1}(A) \to F(A)$ is perfect[17]. The set $F(A)$ has $\mathcal{P}$ so that $g^{-1}(A)$ also has $\mathcal{P}$. On the other hand, the mapping $g \mid g^{-1}(A) : g^{-1}(A) \to A$ is perfect and thus A has $\mathcal{P}$. ■

This lemma is applicable, for instance, to the following properties $\mathcal{P}$: (a) Lindelöf Σ-spaces; (b) paracompact p-spaces; (c) strong Σ-spaces; (d) bi-k-spaces.

3. Separation Axioms and mv-Cleavability

Theorem 3.1 — Let a space X be C-mv-cleavable over the class of normal spaces. Then X is a Hausdorff space.

PROOF : Let $x, y \in X$, $x \neq y$. Choose a normal space Y and a continuous mv-mapping $F : X \to Y$ such that $F^{-1}F(y) = \{y\}$. Then $F(x)$ and $F(y)$ are disjoint closed subsets of Y so that there exist disjoint open sets U and V such that $F(x) \subset U, F(y) \subset V$. The sets $F^{-}(U)$ and $F^{+}(V)$ are open disjoint neighbourhoods of x and y, respectively, so that X is a Hausdorff space. ■

Theorem 3.2 — If a space X is C-closed-mv-cleavable over the class of regular (normal) spaces, then X is also regular (normal).

PROOF : We prove only the normal case. Let A and B be two disjoint closed subsets of X. Take a normal space Y and a continuous closed mv-mapping $F : X \to Y$ such that $F^{-}F(A) = A$. Then $F(A)$ and $F(B)$ are disjoint closed subsets of Y so that there are open sets U and V in Y such that $F(A) \subset U$, $F(B) \subset V$ and $U \cap V = \phi$. The sets $F^{-}(U)$ and $F^{+}(V)$ are open in X witnessing that X is a normal space. ■

There are examples of non-regular (non-normal) spaces which admit continuous single-valued bijections onto regular (normal) spaces, which means that the closedness of mappings in Theorem 3.2 cannot be omitted.

In what follows all the spaces are assumed to be Hausdorff unless otherwise specified.

4. Metric and Generalized Metric Spaces and mv-Cleavability

In this section we generalize some results of Arhangel'skii and others[6,8,9] and Kočinac[23,25,26] concerning single-valued cleavability to the case of mv-cleavability.

Theorem 4.1 — If a space X is perfectly mv-cleavable over the class of metrizable spaces, then X is itseld metrizable.

PROOF : We show first that X is a perfect space. Let K be a closed subset of X. Take a metrizable space Y and a perfect mv-mapping $F : X \to Y$ such that $F^{-}F(K) = K$. The set $F(K)$ is closed in Y so that $F(K) = \bigcap \{V_i : i \in \omega\}$, where V_i

are open subsets of Y. Then $K = F^{+}F(K) = F^{+}(\bigcap \{V_i : i \in \omega\}) = \bigcap \{F^{+} V_i : i \in \omega\}$ is a G_δ-set because the sets $F^{+}(V_i), i \in \omega$, are open. Hence, X is perfect.

Let A be a subset of X. Choose a metrizable space Y and a perfect multi-valued mapping $F : X \twoheadrightarrow Y$ such that $F(X) = Y$ and $F^{-}F(A) = A$. By Proposition 1.3 there exist a space Z and single-valued perfect mappings $g : Z \twoheadrightarrow X$ and $h : Z \twoheadrightarrow Y$ such that $F = h \circ g^{-1}$. The set $F(A)$ is metrizable and thus a paracompact p-space[7]. The property of being a paracompact p-space is both an invariant and inverse invariant under perfect single-valued mappings[7] so that by Lemma 2.7 A is a paracompact p-space. Therefore, every subspace of X is a paracompact p-space. But it is known that a perfect hereditarily paracompact p-space is metrizable[10, 32] and the theorem is proved. ■

Using assertions on the end of section 1 and Theorem 4.1 we obtain :

Theorem 4.1′ — If a compact space X is USC compact-valued mv-cleavable over the class of metrizable spaces, then X is itself metrizable. ■

Let us remark that the following theorem holds.

Theorem 4.2 — If a space X is perfectly mv-cleavable over the class of separable metrizable spaces, then X is also a separable metrizable space.

PROOF : We have to complete the previous proof by : (i) for every $A \subset X$, $g^{-1}(A)$ is a Lindelöf p-space (as a perfect preimage of a separable metrizable space[1, 20]; (ii) the set $A = g(g^{-1}(A))$ is a paracompact p-space and, clearly, a Lindelöf space. So, X is a hereditarily, Lindelöf p-space. By a result of Pytkeev[32] (see also Arhangel'skii[1] and Balogh[10] (which states that the class of hereditarily Lindelöf p-spaces coincides with the class of separable metric spaces), we have that X is a separable metrizable space. ■

Theorem 4.3 — If a GO-space X is perfectly mv-cleavable over the class $\mathcal{P}$ of stratifiable spaces, then X is metrizable.

PROOF : Let A be a subset of X, Y a stratifiable space[15] and $F : X \twoheadrightarrow Y$ a perfect mv-mapping such that $F^{-}F(A) = A$. The set $F(A)$ is stratifiable and so it is a strong Σ-space[11, 15]. A is also a strong Σ-space (see the proof of Theorem 4.5), i.e. X is hereditarily strong Σ-space. It remains to apply this result of Balogh[11] : a GO-space is metrizable iff it is a hereditarily strong Σ-space. ■

Question 4.4 — Is Theorem 4.3 true if $\mathcal{P}$ is the class of semi-stratifiable spaces ? (Recall that a semi-stratifiable GO-space is metrizable).

Theorem 4.5 — If a space X is perfectly mv-cleavable over the class of σ-spaces, then X is also a σ-space.

PROOF : As in the proof of Theorem 4.1 X is a perfect space. Let A be any subset of X, Y a σ-space, F a perfect mv-mapping from X onto Y such that $F^{-}F(A) = A$. The set $F(A)$ is a strong Σ-space (being a σ-space). On the other hand, the last property is preserved by perfect mappings[15] and is also an inverse invariant of such mappings. Applying Lemma 2.7 we obtain that X is a hereditarily strong Σ-space. As X is perfect, one concludes that X is a σ-space by a known result[11, 32]. ■

Theorem 4.6 — If a Tychonoff space X is perfectly mv-cleavable over the class of Moore spaces, then X is also a Moore space.

PROOF : Let A be a subset of X. Choose a Moore space Y and a perfect mv-mapping $F : X \to Y$ such that $F^{-}F(A) = A$. Let Z, g and h be as in Proposition 1.3. A preimage (with completely regular domain) of a Moore space under a perfect mapping is a subparacompact p-space (Burke[15]). Therefore, $g^{-1}(A)$ is such a space. This property is an invariant of perfect mappings, so that A is a subparacompact p-space. As every subparacompact space is θ-refinable[15], X is a hereditarily θ-refinable p-space, hence a Moore space according to a result of Pytkeev[32] (a hereditarily p-space is developable iff it is perfect and θ-refinable). ■

Theorem 4.7 — If a space X is perfectly mv-cleavable over the class of Fréchet-Urysohn spaces, then X is also Fréchet-Urysohn.

PROOF : Every Fréchet-Urysohn space is a hereditarily k-space, a perfect preimage of a k-space is also a k-space and the property of being a k-space is an invariant under closed mappings[1, 7, 17]. Hence, according to Lemma 2.7, X is a hereditarily k-space which means that it is a Fréchet-Urysohn space[7, 17]. ■

Theorem 4.7′ — If a compact space X is USC compact-valued cleavable over the class of Fréchet-Urysohn spaces, then X is also Fréchet-Urysohn. ■

We are going now to give another result on mv-cleavability over the class of Fréchet-Urysohn spaces.

Theorem 4.8 — If a space X is closed C-pointwise mv-cleavable over the class of Fréchet-Urysohn spaces, then X is a Fréchet-Urysohn space.

PROOF : Let $A \subset X$, $x \in \overline{A} \setminus A$. Take an Fréchet-Urysohn space Y and a continuous closed mv-mapping $F : X \to Y$ such that $F^{-}F(x) = \{x\}$. As F is an LSC-mapping one has $F(x) \subset F(\overline{A}) \subset \overline{F(A)}$. On the other hand, $F(x) \bigcap F(A) = \phi$ (otherwise, $F(x) \bigcap F(a) \neq \phi$ for some $a \in A$, i.e. $a \in F^{-}F(x) = \{x\}$, that is $a = x$ which is impossible because $x \notin A$). So, $F(x) \subset \overline{F(A)} \setminus F(A)$. Choose a sequence $(y_i : i \in \omega) \subset F(A)$ which converges to $F(x)$. For every $i \in \omega$ choose a point $x_i \in F^{-}y_i \bigcap A$. Let us show that the sequence $(x_i : i \in \omega)$ converges to x. Let U be a neighbourhood of x. Since F is closed and $\{x\} = F^{-}F(x)$ there exists a neighbourhood V of $F(x)$ such that $F^{-}(V) \subset U$. But (y_i) converges to $F(x)$ so that there is some $k \in \omega$ with $y_i \in V$ for all $i \geq k$. Then $x_i \in F^{-}(V) \subset U$ for each $i \geq k$ which means that (x_i) converges to x. ■

Theorem 4.9 — If a separable space X is perfectly mv-cleavable over the class of bisequential spaces[31], then X is bisequential too.

PROOF : Let A be a subset of X, Y a bisequential space and $F : X \to Y$ a perfect mv-mapping with $F^{-}F(A) = A$. The set $F(A)$ is bisequential so it is a bi-k-space[31]. The last property is a perfect inverse invariant and is preserved by perfect images[31]. Therefore, by Lemma 2.7 X is a hereditarily bi-k-space and so X is ω-bisequential. But every separable ω-bisequential space is bisequential (Arhangel'skii; see, for example, Kočinac[23]). ■

Corollary 4.10 — A separable topological group G which is perfectly mv-cleavable over the class of bisequential spaces is metrizable.

PROOF : By Theorem 4.9 G is a bisequential space. Arhangel'skii has proved that every bisequential topological group is metrizable. ■

5. CARDINAL FUNCTIONS AND mv-CLEAVABILITY

Arhangel'skii[2] has suggested the investigation of cleavable versions of cardinal functions (see references [5], [6], [12], [13], [25], [27], [29] in this connection). Using this idea we define now mv-cleavable cardinal invariants as follows.

Let φ be a cardinal function, τ a cardinal and $\mathcal{F}$ a class of mv-mappings. Denote by $\mathcal{P}(\varphi, \tau)$ the set of all spaces Y such that $\varphi(Y) \leq \tau$. For a space X we define $\mathcal{F}$-cleavable φ as $\min\{\tau : X$ is $\mathcal{F}$-mv-cleavable over $\mathcal{P}(\varphi, \tau)\}$.

It is understood, one of the main problems regarding mv-cleavable versions of cardinal functions is connected with Question B : if a space X is $\mathcal{F}$-mv-cleavable over a class $\mathcal{P}(\varphi, \tau)$, does $\varphi(X) \leq \tau$ hold. In this section we consider such questions.

Let $\mathcal{F}$ denote the class of all USC mappings F with F^- is compact-valued.

Theorem 5.1 — If a (Hausdorff) space X is $\mathcal{F}$-mv-cleavable over the class of all spaces Y of cardinality $\leq \tau$, then $|X| \leq \tau$.

PROOF : Let A be a subset of X. Take a space Y with $|Y| \leq \tau$ and a mapping $F \in \mathcal{F}$, $F : X \to Y$, such that $F^-F(A) = A$. Since $|F(A)| \leq \tau$ and $A = \bigcup \{F^{-1}(y) : y \in F(A)\}$ we have that A is the union of $\leq \tau$ many compact subsets of X. Then by a result from Gerlits *et al.*[19] (if every subset of a Hausdorff space X is the union of $\leq \tau$ many compact sets, then $|X| \leq \tau$) we conclude $|X| \leq \tau$. ■

Theorem 5.2 — If a space X is USC closed mv-cleavable (resp. USC pointwise mv-cleavable) over the class of all spaces Y with $\Psi(Y) \leq \tau$, then $\Psi(X) \leq \tau$ (resp. $\psi(X) \leq \tau$).

PROOF : Let A be a closed subset of X. Fix a space Y with $\Psi(Y) \leq \tau$ and a closed USC mapping $F : X \to Y$ such that $F^-F(A) = A$. The set $F(A)$ is closed in Y so that there exists a collection $\{V_\alpha : \alpha \in \tau\}$ such that $F(A) = \bigcap \{V_\alpha : \alpha \in \tau\}$. Then $A = F^+F(A) = F^+ (\bigcap \{V_\alpha : \alpha \in \tau\}) = \bigcap \{F^+ (V_\alpha) : \alpha \in \tau\}$ and since all sets $F^+ (V_\alpha)$, $\alpha \in \tau$, are open in X (as F is USC) we have the proof. The second case is quite similar. ■

Theorem 5.3 — If a space X is perfectly pointwise mv-cleavable over the class $\mathcal{P}$ of spaces Y such that for every closed $B \subset Y$ we have $\chi(B, Y) \leq \tau$, then $\chi(X) \leq \tau$.

PROOF : Let x be any point in X. Choose a space $Y \in \mathcal{P}$ and a perfect mv-mapping $F : X \to Y$ such that $F^-F(x) = \{x\}$. Let Z, g and h be as in Proposition 1.3 and let $\{V_\alpha : \alpha \in \tau\}$ be a neighbourhood base for $F(x)$ in Y. Since $h : Z \to Y$ is a closed (single-valued) mapping, the family $\{h^{-1} (V_\alpha) : \alpha \in \tau\}$ is a neighbourhood base for $g^{-1}(x) = h^{-1} hg^{-1}(x)$ (see Engelking[17], 2.1. C.(d)). The collection $\{g^{\#} h^{-1}(V_\alpha) : \alpha \in \tau\}$ is a local base for x in X. Indeed, let U be a neighbourhood of x. Then $g^{-1}(U)$ is a neighbourhood of $g^{-1}(x)$ in Y so that there exists $\beta \in \tau$ such that $g^{-1}(x) \subset h^{-1}(V_\beta) \subset g^{-1}(U)$. Hence, $\{x\} = g^{\#} g^{-1}(x) \subset g^{\#} h^{-1}(V_\beta) \subset g^{\#} g^{-1}(U) \subset gg^{-1}(U) = U$. This means $\chi(x, X) \leq \tau$ and since x was an arbitrary point in X then $\chi(X) \leq \tau$. ■

Theorem 5.4 — If a space X is perfectly mv-cleavable over the class of Tychonoff spaces having countable network, then X also has a countable network.

PROOF : Let A be a subset of X. Choose a Tychonoff space Y with $nw(Y) \leq \omega$ and a perfect mv-mapping $F : X \rightarrow Y$ such that $F^{-}F(A) = A$. The set $hg^{-1}(A) = F(A)$ has a countable network, so that it is a Lindelöf Σ-space. Besides, Lindelöf Σ-spaces satisfy the conditions of Lemma 2.7[15]. Therefore, by that Lemma, every subspace of X is a Lindelöf Σ-space. By a result of Hodel[20], then X has a countable network. ■

We are going now to prove one general result connected with cardinal functions and mv-cleavability.

Theorem 5.5 — Let τ be an infinite cardinal number and $\mathcal{P}$ a hereditary and τ-multiplicative class of topological spaces. Let $\{A_\alpha : \alpha \in S\}$ be a family of pairwise disjoint subsets of a space X such that $|A_\alpha| \leq 2^\tau$ for every $\alpha \in S$. If X is LSC (USC compact-valued, C compact-valued) mv-cleavable over $\mathcal{P}$, then there exists a LSC (USC compact-valued, C compact-valued) mapping Φ from X onto a space $Z \in \mathcal{P}$ such that $\Phi | A_\alpha$ is 1-1 for every $\alpha \in S$.

PROOF : Let us consider the LSC case. For every $\alpha \in S$ there is a point separating family γ_α of subsets of A_α such that $|\gamma_\alpha| \leq \tau$, say $\gamma_\alpha = \{A_{\alpha,\beta} : \beta \in \tau\}$ (see, for instance, Arhangel'skii and coworkers[7, 8]). Let $X_\beta = \bigcup\{A_{\alpha,\beta} : \alpha \in S\}$, $\beta \in \tau$. For each $\beta \in \tau$ fix a space $Y_\beta \in \mathcal{P}$ and a LSC mapping $F_\beta : X \rightarrow Y_\beta$ such that $F_\beta^{-} F_\beta(X_\beta) = X_\beta$. It is known[17] that the Cartesian product $\Phi = \Pi\{F_\beta : \beta \in \tau\}$ defined by $\Phi(x) = \Pi\{F_\beta(x) : \beta \in \tau\}$, $x \in X$, is a LSC mapping from X onto the space $Z = \Phi(X) \subset \Pi\{Y_\beta : \beta \in \tau\} \in \mathcal{P}$. It remains to check that $\Phi | A_\alpha$ is 1-1. Take two distinct points $p, q \in A_\alpha$. There exists $\beta \in \tau$ such that $p \in A_{\alpha,\beta}, q \notin A_{\alpha,\beta}$, and thus $F_\beta(p) \cap F_\beta(q) = \phi$, hence $\Phi(p) \cap \Phi(q) = \phi$.

For other two cases one should observe that the Cartesian product of USC compact-valued mappings is a USC compact-valued mapping[17]. ■

Corollary 5.6 — Let $\mathcal{P}$ be as in Theorem 5.5. If a space X of cardinality $\leq 2^\tau$ is LSC (USC compact-valued, C compact-valued) mv-cleavable over $\mathcal{P}$, then X admits a 1-1 LSC (USC compact-valued, C compact-valued) mapping onto a space $Y \in \mathcal{P}$. ■

For a space X we define :

$Iw(X) = \min\{w(Y)$: there exists a continuous mv-injection from X onto $Y\}$.

The cardinal number $Iw(X)$ is called the 'I-weight' of X. Of course, one has $Iw(X) \leq iw(X)$ for every space X.

Theorem 5.7 — Let X be a space with $e(X) \leq \tau$ or $cL(X) \leq \tau$. If X is C compact-valued cleavable over the class $\mathcal{P}$ of spaces Y such that $nw(Y) \leq 2^{\tau}$ and $\Psi(Y) \leq \tau$, then $Iw(X) \leq 2^{\tau}$.

PROOF : Let A be a subset of X. Choose a space $Y \in \mathcal{P}$ and a continuous compact-valued mapping $F: X \to Y$ such that $F^{-}F(A) = A$. From $|Y| \leq nw(Y)^{\psi(Y)} \leq (2^{\tau})^{\tau} = 2^{\tau}$ (Hodel[21]) it follows $|F(A)| \leq 2^{\tau}$. For every $y \in Y, F^{-1}(y)$ is closed in X so that $e(F^{-1}(y)) \leq e(X)$ (resp. $cL(F^{-1}(y)) \leq cL(X) \leq \tau$) and consequently, $e(A) = e(\bigcup\{F^{-1}(y) : y \in F(A)\}$ (resp. $cL(A) = cL(\bigcup\{F^{-1}(y) : y \in F(A)\})$ $\leq 2^{\tau}$. Therefore, we have $he(X) \leq 2^{\tau}$ (resp. $hcL(X) \leq 2^{\tau}$). But $he(X) = s(X)$ (Hodel[21]) [resp. $s(X) \leq hcL(X)$, as can be easily verified (Bella *et al.*[12], Kočinac[27])]. On the other hand, $\psi(X) \leq \tau$ (Theorem 5.2), so that $|X| \leq 2^{s(X)\psi(X)} \leq 2^{2^{\tau}}$ (Hodel[21]). The product of $\leq 2^{\tau}$ many spaces from $\mathcal{P}$ has the netweight $\leq 2^{\tau}$ so that by Corollary 5.6 there are a space Z with $nw(Z) \leq 2^{\tau}$ and a continuous compact-valued mv-injection Φ from X onto Z. Besides, there is a continuous single-valued bijection $f: Z \to T$ from Z onto a space T with $w(T) \leq 2^{\tau}$. The composition $f \cdot \Phi : X \to T$ is a continuous mv-injection which witnesses $Iw(X) \leq 2^{\tau}$. ■

Corollary 5.8 — Let X be a space with $e(X) \leq \tau$ or $cL(X) \leq \tau$. If X is C compact-valued cleavable over the class $\mathcal{P}$ of spaces Y such that $nw(Y) \leq 2^{\tau}$, $\Psi(Y) \leq \tau$ and for every $\mathcal{P}' \subset \mathcal{P}$ with $|\mathcal{P}'| \leq 2^{\tau}$, $\mathcal{K}(\Pi\{Y : Y \in \mathcal{P}'\}) \leq 2^{\tau}$, then $|X| \leq 2^{\tau}$. ■

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