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METRICALLY SEPARABLE AND APPROXIMATELY CONTINUOUS FUNCTIONS

M. C. CHAKRABORTY

Department of Mathematics, University of Kalyani, Kalyani 741 235, West Bengal

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In this paper we have shown that if a real function $f(x)$ defined on a linear set E, be approximately continuous almost everywhere in E, then it is metrically separable relatively to E.

1. INTRODUCTION

Jeffery[5] has defined metrically separated sets (p. 52) and mètrically separable functions (p. 190) in real number space. These concepts are so rich that they have been extended recently by others in different abstract spaces. Theorem 7.6 (Jeffery[5], p. 190) states that if a real function is metrically separable relatively to a linear set E, then at almost all points of E the function is approximately continuous over E. It appears that whether converse of the above theorem holds, remains an unanswered outstanding problem. Our main objective is to prove that the converse of this theorem holds. The proof is done by considering an alternative definition of separable function[2] concurrently with that of Jeffery. The method can be extended in other abstract spaces also. As an immediate consequence we get equivalence of the two definitions of separability of a function. Like the classical result of Blumberg[1] on measurable boundaries of an arbitrary function, we obtain properties of upper (lower) approximate limit functions[2], similar to those of measurable cover functions introduced by Eames and May[3].

2. KNOWN RESULTS

Although known, the definitions and results of this section are included for completeness. $f(x)$ is a real valued function on a linear set E, $m*$ denotes outer Lebesgue measure, and m the Lebesgue measure. $\bar{D}^*$) denotes upper outer density function.

Definition 2.1 (Jeffery[5], p. 52) — Two subsets A, B of E are said to be metrically separated, if for $\varepsilon > 0$ arbitrary, there exist open sets O_1 and O_2 such that $O_1 \supset A, O_2 \supset B$ and $m(O_1 \cap O_2) < \varepsilon$.

Definition 2.2 (Jeffery[5], p. 190) — $f(x)$ is said to be metrically separable relatively to E if for every real number c, the sets

$$\{x \in E : f(x) < c\} \quad \text{and} \quad \{x \in E : f(x) \geq c\}$$

are metrically separated.

An alternative definition is the following :

Definition 2.3 (Bose[2], Definition 2.3) — $f(x)$ is said to be metrically separable relatively to E, if for every real number c, the sets

$$\{x \in E : f(x) \leq c\} \quad \text{and} \quad \{x \in D : f(x) > c\}$$

are metrically separated.

Definition 2.4 — A point x is called a d-limit point[4] of a set $F \subset E$, if $\overline{D}^*(F, x) > 0$. The set F is said to be d-closed in E if it contains all its d-limit points in E. The set of all d-limit points of E is denoted by E'.

Definition 2.5 (Bose[2], Definition 1.4) — Let x be a d-limit point of E. The infimum [supremum] of all real numbers k for which

$$\overline{D}^* (\{y \in E : f(y) \leq k\}, x) > 0 \ [\overline{D}^* (\{y \in E : f(y) \geq k\}, x) > 0]$$

is called the lower [upper] approximate limit of $f(y)$ at x and is denoted by $\underline{f}(x)$ $[\overline{f}(x)]$.

Definition 2.6 (Bose[2], Definition 1.5) — Let $x \in E$ be a d-limit point of E. $f(x)$ is said to be lower [upper] approximately continuous at x if

$$\underline{f}(x) \geq f(x) \ [\overline{f}(x) \leq f(x)].$$

It is said to be approximately continuous at the point x if it is both lower and upper approximately continuous at that point.

Theorem 2.1 (Bose[2], Theorem 2.1) — $f(x)$ is lower [upper] approximately continuous at a d-limit point $c \in E$, if and only if to every $\varepsilon > 0$ there corresponds a subset of E having outer density zero at c, such that

$$f(x) > f(c) - \varepsilon \ [f(x) < f(c) + \varepsilon]$$

except for points in this subset.

Note 2.1 : From the above theorem it follows that, if $f(x)$ is approximately continuous at a point of outer density of E according to Definition 5.4 (Jeffery[5], p. 118), then $f(x)$ is so according to Definition 2.6.

Theorem 2.2 (Bose[2], Theorem 2.2) — $f(x)$ is lower [upper] approximately continuous at each point of E, if and only if for every real number c, the set

$$\{x \in E : f(x) \leq c\} \quad [\{x \in E : f(x) \geq c\}]$$

is d-closed in E.

Theorem 2.3 (Bose[2], Theorem 2.3) — The function $\underline{f}(x)$ is lower approximately continuous on $E \cap E'$ (i.e., at every point of $E \cap E'$).

3. Main Theorems

Theorem 3.1 — If $f(x)$ is approximately continuous almost everywhere in E, then it is metrically separable relatively to E.

Proof : Let $f(x)$ be approximately continuous over E according to Jefferey's definition at every point of $E_1 \subset E$ where $m^*(E - E_1) = 0$. Then at any point of E_1, $f(x)$ is both lower and upper approximately continuous over E_1. For any real number c, let

$$A_1 = \{x \in E_1 : f(x) < c\} \quad \text{and} \quad A_2 = \{x \in E_1 : f(x) \geq c\},$$

$$A_3 = \{x \in E_1 : f(x) \leq c\} \quad \text{and} \quad A_4 = \{x \in E_1 : f(x) > c\}.$$

By Theorem 2.2, the sets A_2 and A_3 are d-closed in E_1. So, by Theorem 5.4 (Jeffery[5], p. 116), the sets A_1 and A_2, as well as the sets A_3 and A_4 are metrically separable relatively to E_1 according to both the Definitions 2.2 and 2.3. Since $m^*(E - E_1) = 0$, $f(x)$ is metrically separable relatively to E. The above method of proof obviously holds if $f(x)$ is approximately continuous a.e. in E according to Definition 2.6. This completes the proof.

Since Theorem 7.6 (Jeffery[5], p. 190) also holds if we start with Definition 2.3 instead of Definition 2.2, applying that theorem and Theorem 3.1 successively we get the following :

Theorem 3.2 — $f(x)$ is metrically separable relatively to E according to Definition 2.3, if and only if it is so according to Definition 2.2.

By virtue of Theorem 3.1, the statement of Theorem 7.17 (Jeffery[5], p. 200) assumes the following nice form :

Theorem 3.3 — If $f(x)$ is approximately continuous almost everywhere in E, then the extreme approximate derived numbers of $f(x)$ over E are approximately continuous almost everywhere in E.

4. Upper and Lower Approximate Limit Functions

Theorem 3.2 has wide application in getting the results of this section and the next.

Theorem 4.1 — $\underline{f}(x)$ and $\bar{f}(x)$ are separable relatively to $E \cap E'$.

Proof : The theorem follows by Theorem 2.3 and a corresponding result for $\bar{f}(x)$.

Theorem 4.2 — $\underline{f}(x) \leq f(x)$ and $\bar{f}(x) \geq f(x)$ a.e. in $E \cap E'$. The proof of the theorem is not difficult and is so omitted.

Theorem 4.3 — If functions $h(x)$ and $g(x)$ are defined on E and are separable relatively to E and if $h(x) \leq f(x) \leq g(x)$ a.e. in E, then

$$h(x) \leq \underline{f}(x) \text{ and } \bar{f}(x) \leq g(x) \text{ a.e. in } E \cap E'.$$

PROOF: We prove the first part only. We may assume that $\underline{f}(x) < \infty$ for $x \in E \cap E'$. Let $r_1, r_2, ..., r_n, ...$ be an enumeration of the set of rational numbers. Write

$$P_n = \{x \in E \cap E' : h(x) > r_n > \underline{f}(x)\},$$

$$R_n = \{x \in E : h(x) \leq r_n\}$$

and

$$E_n = \{x \in E : f(x) \leq r_n\}.$$

For any $x \in P_n$, $\overline{D}^* (E_n, x) > 0$ and so, $\overline{D}^*(R_n, x) > 0$. Since P_n and R_n are separated, we obtain $m^* (P_n) = 0$ and so, $h(x) \leq \underline{f}(x)$ a.e. in $E \cap E'$.

Note 4.1 : In view of Theorem 4.3, $\underline{f}(x)$ and $\bar{f}(x)$ may be called lower and upper separable cover functions of $f(x)$ on $E \cap E'$. They have properties similar to those of measurable cover functions[3] of a real valued function and their proofs are also similar.

§ 5. In this section we assume that every point of E is a d-limit point of E.

Definition 5.1 — Let $f(x)$ and $g(x)$ be two real functions defined on E. The local distance of f and g relatively to E at a point x is

$$\overline{D}^* (\{y \in E : f(y) \neq g(y)\}, x)$$

and is denoted by $d_x (f, g)$.

We write $f \in A_x$, if there is a function g, separable relatively to E, such that $d_x (f, g) = 0$.

Theorem 5.1 — $f \in A_x$ if and only if either $d_x (f, \underline{f}) = 0$, or $d_x (f, \bar{f}) = 0$.

The proof of the above theorem is similar to that of Theorem 4 of Eames and May[3] and is so omitted.

REFERENCES

1. H. Blumberg, *Acta Math.* **65** (1935), 263-82.
2. M. K. Bose, *Indian J. Mech. Math.* **15** (1977), 50-58.
3. W. Eames and L. E. May, *Canad. Math. Bull.* **10** (1967), 519-23.
4. C. Goffman and D. Waterman, *Proc. Am. Math. Soc.* **12** (1961), 116-21.
5. R. L. Jeffery, *The Theory of Functions of a Real Variable*, Toronto, 1953.