Indian J. pure appl. Math., 22(1): 13-21, January 1991

ON MORDELL'S EQUATION $y^2 - k = x^3$

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(*Received* 11 *October*, 1989; *accepted* 7 *August* 1990)

1. INTRODUCTION

In his doctoral dissertation, Mohanty[3] solved all equations $y^2 - k = x^3$, $101 \leq k \leq 200$ with incomplete solutions for $k = 120, 130, 193$ and 198. In the present paper we complete these four cases.

2. THEORETICAL PART

We quote the theorems we will use in this paper.

Theorem H1[2] — If $2f$ does not contain primes which divide into two distinct prime ideals in the quadratic field $Q(\sqrt{k})$, then all the integral solutions of the equation $y^2 - kf^2 = x^3$ are contained in the $\frac{1}{2}(3^{e+1}+1)$ different equations

$$N(\beta_i)\ (\pm\ y + f\sqrt{k}) = \epsilon\ \beta_i\ a^3$$

where (β_i), $i = 1, 2, \ldots, 3^e$ is the cube of an arbitrary ideal, one from each of the 3^e classes C_i with the property that $C_i^3 = (1)$, a is an integer in $Q(\sqrt{k})$, and e the basis number for 3 in the group of ideal classes.

Theorem H2[2] — If $2f$ contains r distinct primes p_i, $i = 1, 2, \ldots, r$, which divide into two distinct prime ideals in the field $Q(\sqrt{k})$ and if the class number $h(Q(\sqrt{k}))$ is not divisible by 3, then all the integral solutions of the equation $y^3 - kf^2 = x^3$ are contained in the equations

$$\prod_{i=1}^{r} p_i^{q_i}\ (\pm\ y + f\sqrt{k}) = \prod_{i=1}^{r} p_i^{h_i}\ a^3 = \epsilon\ \beta\ a^3$$

where $h_i = 0$ or the least positive integer making $p_i^{h_i}$ a principal ideal, and all the combinations of these values are considered. If $h_i = 0$ we put $q_i = 0$ and if $h_i > 0$ (and hence not divisible by 3) we put $q_i = h_i - 2$ if $h_i \equiv 1 \pmod 3$ and

$q_i = h_i - 1$ if $h_i = 2 \pmod 3$. Here a is an integer in $Q(\sqrt{k})$ and $\epsilon = 1, \epsilon_0$ or ϵ_0. The number of different equation is $\frac{1}{2}(3^{r+1} - 1)$ if $k > 1$ of $k = -3$, and $\frac{1}{2}(3^r + 1)$ if $k < 0$, $k \neq -1$ or -3. The case $k = -1$ yields $3^r + 1$ equations.

Theorem H3[3] – Let $F(u, v) = (1, P, Q, R) = 1$. Suppose $F(\theta, -1) = 0$ and let $\epsilon = a\theta^2 + b\theta + c$ be the fundamental unit of $Z[\theta]$. Put $(a, b) = d$. Let p be an odd prime divisor of $k = N(-a\theta + b + ap)/d^2$, and let $\epsilon^m = a_n\theta^2 + bm\theta + c_m$ be the least power of ϵ such that $a_m \equiv b_m \equiv O \pmod p$. Then if $a_n \not\equiv O \pmod{p^2}$, the equation $\epsilon^n = u + v\theta$ is impossible for $n \neq 0$.

Theorem M1[4] – If $\epsilon = a\theta^2 + b\theta + c$ is a unit in $Z[\theta]$. Put $(a, b) = d$. Let p be an odd prime divisor of $k = N(-a\theta + b + ap)/d^2$, and let $\epsilon^m = a_n\theta^2 + bm\theta + c_m$ be the least power of ϵ such that $a_m \equiv b_m \equiv 0 \pmod p$. Then if $a_n \not\equiv 0 \pmod{p^2}$. the equation $\epsilon^n = u + v\theta$ is impossible for $n \neq 0$.

Theorem M1[4] – If $\epsilon = a\theta^2 + b\theta + c$ is a unit in $Z[\theta]$ and $p^\alpha \| a$, $p^B \| b$ where p is a prime, then $\epsilon^n = u + v\theta$ is impossible for $n \neq 0$ in the following cases:

(i) when $1 \leq \alpha \leq \beta$ and p is odd,

(ii) when $2 \leq \alpha \leq \beta$ and p is 2,

(iii) when $\beta \leq \alpha < 2\beta$ and p is odd,

(iv) when $\beta \leq \alpha < 2\beta - 1$ and p is 2.

Theorem M2[4] – Let $\epsilon = a_1\theta^2 + b_1\theta + c_1$ be a unit in $Z[\theta]$, where $\theta^3 - p_1\theta - q_1 = 0$. If $p_1 \equiv 0$. If $p_1 \equiv O \pmod 3$, then $\epsilon^n = u + v\theta$ is impossible for $n \neq 0$ provided $a \not\equiv O \pmod 3$, $b_1^2 + 2a_1c_1 \not\equiv O \pmod 3$, and $b_1^2 c_1 + a_1c_1^2 + a_1^2b_1 q_1 \not\equiv O \pmod 3$.

3. Solution of the Cases $k = 120, 130, 193, 198$

(A) $y^2 - 120 = x^3$.

The fundamental unit of $Q(\sqrt{30})$ is $\eta = 11 + 2\sqrt{30}$ and $h(Q(\sqrt{30})) = 2$. Then, by Theorem $H1$, we must consider the following two equations for all the integer solutions of $y^2 - 120 = x^2$.

$$+ y + 2\sqrt{30} = (a + b\sqrt{30})^3 \qquad \ldots(1)$$

$$+ y + 2\sqrt{30} = (11 + 2\sqrt{30})(a + b\sqrt{30}). \qquad \ldots(2)$$

The first equation yields $3a^2b + 30b^3 = 2$ which is impossible (mod 3). The second one yields

$$2a^3 + 33a^2b + 180ab^2 + 330b^3 = 2. \qquad \ldots(3)$$

From (3) we see that $ab \equiv O \pmod 2$ and a and b are not both even. Suppose b is even. Then the substitution $a = u - 11v$, $b = 2v$ in (3) gives

$$F(u, v) = u^3 - 3uv^2 + 22v^3 = 1.$$

This corresponds to the ring $Z[\theta]$, where $\theta^3 - 3\theta - 22 = 0$, with the fundamental unit $\epsilon = -13322893\theta^2 - 2556967\theta + 140877207 = a_1\theta^2 + c_1$. We must solve

$\epsilon^n = u + v\theta$. Here we have $p_1 \equiv O \pmod 3$, $q_1 \equiv 1 \pmod 3$, $a_1 \equiv 2 \pmod 3$, $b_1 \equiv 2 \pmod 3$ and $c_1 \equiv O \pmod 3$. Then by Theorem $M2$, $n = 0$ is the only solution of $\epsilon^n = u + v\theta$. Hence $F(u, v) = (1, 0, -3, 22) = 1$ has only one solution which corresponds to $x = 1$, $y = \pm 11$.

Next we consider $a \equiv O \pmod 2$. The substitution $a = 2u - 6v$, $b = v$ in (3) yields

$$F(u, v) = 8u^3 - 6u^2v + 3v^3 = 1. \qquad \ldots(4)$$

Let $u = u_1$ and $v = v_1/3$. Then we have

$$v_1^3 - 18v_1u_1^2 + 72u_1^3 = 9. \qquad \ldots(5)$$

Consider the field $Q(\theta)$ where $\theta^3 - 18\theta - 72 = 0$. We have $|\Delta| = N(3\theta^2 - 18) = 27N(\theta^2 - 6) = 2^5 \cdot 3^6 \cdot 5$. The only possible fractional integers are $\theta/2$, $\theta^2/2$, $\theta/3$ and $\theta^2/3$. We find that $\theta/2$ and $\theta/3$ are not integers, but $\theta^2/2$ and $\theta^2/3$ are. So an integral basis is $[1, \theta, \frac{1}{6}\theta^2]$. The equation of $\frac{1}{6}\theta^2 = \lambda$ is $\lambda^3 - 6x^2 + 9x + 24 = 0$. We see that $\lambda/3$ and $\lambda^2/3$ are not integers and $\lambda^3 \equiv O(\bmod 3)$. So

$$3 = [3, \lambda]^3 = [3, \lambda, \lambda^2]^3 = \left[3, \frac{\theta^2}{6}, \frac{\cdot\theta^2 + 4\theta}{2}\right].$$

By Voronoi's method[6] we find $N\left(\dfrac{-81 + 9\theta + \theta^2}{3}\right) = 3$ and

$\epsilon = 9305 - 1517\theta - 28\theta^2$. Since $N(v_1 + u_1\theta^2) = q$ we must solve

$$(v_1 + u_1\theta) = \left(\frac{-81 + 9\theta + \theta^2}{3}\right)(9305 - 1507\theta - 28\theta^2)^n$$

$$= (873 - 118\theta - 7\theta^2)(a_n + b_n\theta + c_n\theta^2). \qquad \ldots(6)$$

The coefficient of θ^2 in the right side of (6) is $-7a_n - 118b_n + 747c_n$ and we look for an n such that $-7a_n - 118b_n + 747c_n = 0$.

By Theorem $H3$ we look at $|N(-1517 - 9305\theta) = N(1517 + 9305\theta)$ and see that it is divisible by 7459. Since the period of ϵ mod 7459 is 7458 and $-7a_n - 118b_n + 747c_n \not\equiv O \pmod{7459}$ for $0 \le n \le 7458$, the equation (6) is impossible and so is (4). Hence $y^2 - 120 = x^3$ has only one solution given by $(x, y) = (1 \pm 11)$.

(B) $y^2 - 130 = x^3$.

The fundamental unit of $Q(\sqrt{130})$ is $\eta = 57 + 5\sqrt{130}$ and $h(Q(\sqrt{130})) = 4$. Therefore, by Theorem $H1$, all the integral solutions of $y^2 - 130 = x^3$ can be obtained from

$$\pm y + \sqrt{130} = (a + b\sqrt{130})^3$$

and

$$\pm y + \sqrt{130} = (57 + 5\sqrt{130})(a + b\sqrt{130})^3.$$

The above equations yield

$$3a^2b + 130b^3 = 1 \qquad \ldots(1)$$

and

$$5(a^3 + 3 \cdot 130ab^2) + 57(3a^2b + 130b^3) = 1 \qquad \ldots(2)$$

respectively. Clearly (1) has no solution in integers. The substitution $a = u - 12v$,

$b = v$ in (2) gives $F(u, v) = 5u^3 - qu^2v + 6uv^2 - 6v^3 = 1$. Now let $u = u_1/5$, $v = v_1$. Then we have $u_1^3 - qu_1^2v_1 + 30u_1v_1^2 - 150v_1^3 = 25$. Letting $u_1 = u_2 + 3v_2$ and $v_1 = v_2$ we get $u_2^3 + 3u_2v_2^2 - 114v_2^3 = 25$. This corresponds to the ring $Z[\theta]$ where $\theta^3 + 3\theta + 114 = 0$. We have $\Delta = N(3\theta^2 + 3) = 3^3N(\theta^2 + 1) = 3^3 \cdot 2^3 \cdot 5^3 \cdot 13$. Clearly $\theta/3$ and $\theta^2/3$ are not integers. We see that $\theta^3 + 3\theta + 114 \equiv \theta(\theta + 1)^2 \pmod 2$ and $(\theta + 1)(\theta + 2)^2 \pmod 5$. We find that $(\theta^2 + \theta)/2$ is not an integer but $\dfrac{(\theta + 1)(\theta + 2)}{5} = \dfrac{2 + 3\theta + \theta^2}{5}$ is an integer. So an integral basis is $\left[1, \theta, \dfrac{2 + 3\theta + \theta^2}{5}\right]$.

In the field $Q(\theta)$, $\epsilon = \dfrac{1}{5}(3115747939 + 1530972371\theta + 185208057\theta^2)$. Clearly S divides the index of θ. Now we compute the index of $\alpha = \dfrac{2 + 3\theta + \theta^2}{5}$ using results from p. 112 of Delone and Faddev[1]. We have $a = 5$, $t = -2$, $S = 1$, $b = -6$, $c = 3$, $d = 4$ and $5 \nmid f(0, 1)$. The equation of α is $\alpha^3 + 42\alpha - 88 = 0$. The index of α is 4 which is not divisible by 5. Since $\alpha^3 + 42\alpha = 88 \equiv (\alpha + 1)^2(\alpha + 3) \pmod 5$, $5 = [5, (4 + \alpha)]^2 [5, 3 + \alpha]$. Therefore there are three ideals of norm 25.

In the process of finding ϵ we find three non-associated integers with norm 25. They are

(1) $\dfrac{1}{5}(4677569 + 1761471\theta + 162397\theta^2)$,

(2) $11953585 + 1982092\theta - 127644\theta^2$, and

(3) $-22979 + 35702\theta + 8756\theta^2$.

Now letting $\epsilon^n = a_n + b_n\theta + c_n\theta^2$ we have to solve $u + v\theta = I_i\epsilon^n$ $(i = 1, 2, 3)$ where I_i is an integer of norm 25. The coefficient of θ^2 in the right sides are respectively $E_1 = 162397a_n + 1761471b_n + 4190378c_n$, $E_2 = -127644a_n + 1982092b_n + 12336517c_n$. and $E_3 = 8756a_n + 35702b_n - 49247c_n$. We see that $E_1 = 0$ is impossible (modulo 29), $E_2 = 0$ is impossible modulo 186 and finally $E_3 = 0$ is impossible modulo 174.

Thus $y^2 - 130 = x^3$ has no integer solution.

(C) $y^2 - 193 = x^3$.

The fundamental unit of $A(\sqrt{193})$ is $\eta = 1764132 + 126985\sqrt{193}$ and $h(Q(\sqrt{193})) = 1$. We have $4\epsilon = (125 + 9\sqrt{193})(125 - 9\sqrt{193})$ where ϵ is an unit. Then, by theorem $H2$, all the solutions of $y^2 - 193 = x^3$ can be obtained from the following five equations:

$$\pm y + \sqrt{193} = \left(\frac{a + b\sqrt{193}}{2}\right)^3$$

$$\pm y + \sqrt{193} = (1764132 + 126985\sqrt{193})\left(\frac{a + b\sqrt{193}}{2}\right)^3$$

$$\frac{1}{2}(\pm y + \sqrt{193}) = \left(\frac{125 + 9\sqrt{193}}{2}\right)\left(\frac{a + b\sqrt{193}}{2}\right)^3$$

$$\frac{1}{2}(\pm y + \sqrt{193}) = \left(\frac{125 + 9\sqrt{193}}{2}\right)$$

$$(1764132 + 126985\sqrt{193})\left(\frac{a + b\sqrt{193}}{2}\right)^3$$

$$\frac{1}{2} = (\pm y + \sqrt{193}) = \left(\frac{125 + 9\sqrt{193}}{2}\right)$$

$$(1764132 - 126985\sqrt{193})\left(\frac{a + b\sqrt{193}}{2}\right)^3.$$

The equations yield respectively

$$3a^2 + 193b^3 = 8, \quad \ldots(1)$$

$$126985(a^3 + 3 \cdot 193ab^2) + 1764132(3a^2b + 193b^3) = 8, \quad \ldots(2)$$

$$9(a^3 + 3 \cdot 193ab^2) + 125(3a^2b + 193b^3) = 8, \quad \ldots(3)$$

$$31750313(a^3 + 3 \cdot 193ab^2) + 44108945(3a^2b + 193b^3) = 8, \quad \ldots(4)$$

$$4063(a^3 + 3 \cdot 193ab^2) - 56445(3a^2b + 193b^3) = 8. \quad \ldots(5)$$

Clearly eqn. (1) is insoluble in integers. The equaion (3) is impossible modulo 9 and the equation (4) is impossible modulo 7. From eqn. (2) it is clear that a and b are both even. The substitution $a = -250u + 778v$, $b = 18u - 56v$ in gives

$$F(u, v) = 4u^3 + 6u^2v - 3uv^2 + 5v^3 = 1.$$

Let $u = \frac{u_1}{2}$, $v = v_1$. Then $u_1^3 + 3u_1^2 - 3u_1^2v_1 + 10v_1^3 = 2$. Finally let $u_1 = u_2 - v_2$, $v_1 = v_2$. Then $u_2^3 - 6u_2v_2^2 + 15v_2^3 = 2$. This corresponds to the ring $Z[\theta]$ defined by $\theta^3 - 6\theta - 15 = 0$. Here $|\Delta| = 27 \cdot 193$. Since $\frac{\theta^2}{3} \notin Z[\theta]$, $[1,\theta,\theta^2]$ is an integral basis. We have $\epsilon = -16391 - 4750 + 3504\theta^2$ and $\theta^3 - 6\theta - 15 \equiv \theta^3 + 1 = (\theta + 1)(\theta^2 - \theta + 1) \pmod 2$. So $2 = \mathcal{P}_2\mathcal{P}_2'$. There in only one ideal of norm 2. Since $N(-76 - 19\theta + 13\theta^2) = 2$, we have to solve

$$U + v\theta = (-76 - 19\theta + 13\theta^2)\epsilon_0^n = (-76 - 19\theta + 13\theta^2)(a_n + b_n\theta + c_n\theta^2).$$

The coefficient of θ^2 on the right side is $13a_n - 19b_n + 2c_n$. We look for an n so that $13a_n - 19b_n + 2c_n = 0$. Since $\epsilon_0 \equiv 1 \pmod 2$ we have $\epsilon_0^n \equiv 1 \pmod 2$. Therefore $a_n \equiv 1 \pmod 2$ while b_n and c_n are congruent to $O \pmod 2$. Hence $13a_n - 19b_n + 3c_n = 0$ is impossibe (mod 2). Thus (6) is impossible.

The substitution $a = -125a_2 + 14b_2$ and $b = -9a_2 + b_2$ inb (5) gives

$$5b_2^3 + 6b_2^2a_2 - 12b_2a_2^2 + 40a_2^3 = 8. \quad \ldots(7)$$

Then b_2 is even. We put $b_2 = 2u$, $a_2 = v$ in (7), obtaining

$$F(u, v) = 5u^3 + 3u^2v - 3uv^2 + 5v^3 = 1. \quad \ldots(8)$$

The form (5, 3, - 3, 5) defines the ring $R(1, \alpha, \beta)$ where

$$\alpha^3 - 3\alpha^2 - 15\alpha - 125 = 0, \beta^3 + 3\beta^2 + 15\beta - 125 = 0 \text{ and } \alpha\beta = 25.$$

Then $\beta = \dfrac{25}{\alpha} = \dfrac{-15 - 3\alpha + \alpha^2}{5}$. Let $\alpha = \theta + 1$. Then $\theta^3 - 18\theta - 142 = 0$ and $\beta = \dfrac{-17 - \theta + \theta^2}{5}$. So $R(1, \alpha, \beta) \equiv R\left[1, \theta + 1, \dfrac{-17 - \theta + \theta^2}{5}\right]$ $\equiv R\left[1, \theta, \dfrac{3 + 4\theta + \theta^2}{5}\right]$ and this is also an integral basis for $Q(\theta)$. Then $R[a, \alpha, \beta] \equiv Z[\theta]$. We have to check whether $[5, \alpha] \equiv [5, \theta + 1]$ is a principal ideal in $Z[\theta]$. By Voronoi's method we find this is not the case and thus (B) is no soluble. Thus $y^2 - 193 = x^3$ is insoluble.

(D) $y^2 - 198 = x^3$.

The fundamental unit of $Q(\sqrt{22}$ is $\eta = 197 + 42\sqrt{22}$ and $h(Q(\sqrt{22})) = 1$. Here 3 splits into two different prime ideal in $Q(\sqrt{22})$. Therefore by Theorem *H2*, all the integer solutions of $y^2 - 198 = x^3$ can be obtained from the following five equations:

$$\pm y + 3\sqrt{22} = (a + b\sqrt{22})^3$$

$$\pm y + 3\sqrt{22} = (197 + 42\sqrt{22})(a + b\sqrt{22})^3$$

$$\pm \frac{y}{3} + \sqrt{22} = (5 + \sqrt{22})(1 + b\sqrt{22})^3$$

$$\pm \frac{y}{3} + \sqrt{22} = (5 + \sqrt{22})(197 + 42\sqrt{22})(1 + b\sqrt{22})^3$$

$$\pm \frac{y}{3} + \sqrt{22} = (5 + \sqrt{22})(197 - 42\sqrt{22})(a + b\sqrt{22})^3.$$

These yield, respectively

$$3a^2b + 22b^3 = 3, \qquad \ldots(1)$$

$$42(a^3 + 3 \cdot 22ab^2) + 197(3a^2b + 22b^3) = 3, \qquad \ldots(2)$$

$$(a^3 + 3 \cdot 22ab^2) + 5(3a^2b + 22b^3) = 1, \qquad \ldots(3)$$

$$407(a^3 + 3 \cdot 22ab^2) + 1909(3a^2b + 22b^3) = 1, \qquad \ldots(4)$$

and

$$-13(a^3 + 3 \cdot 2ab^2) + 61(3a^2b + 22b^3) = 1. \qquad \ldots(5)$$

From (1), we have $b \equiv O \pmod 3$. Then the left side of (1) $\equiv O \pmod 9$ and the right side of (1) $\equiv 3 \pmod 9$. Hence (1) is impossible.

The substitution $a = u - 5v$, $b = v$ in (3) gives

$$F(u, v) = u^3 - 9uv^2 + 30v^3 = 1. \qquad \ldots(6)$$

This corresponds to the ring $Z[\theta]$, where $\theta^3 - 9\theta - 30 = 0$, with fundamental unit

$$\epsilon_0 = 9298969580\theta^2 + 1340415158439\theta - 5581745457733 = a_1\theta^2 + b^1\theta + c_1.$$

Since $a_1 \equiv b_1 \equiv O \pmod 3$ and $a_1 \not\equiv O \pmod{3^2}$, by Theorem $M1$, $n = 0$ is the only solution for $\epsilon_0^n = u + v\theta$. Hence, $f(u, v) = u^3 - 9uv^2 + 30v^3 = 1$ has only one solution, which corresponds to $x = 3$, $y = \pm 15$.

The substitution $a = 19u + 119v$, $b = -4u - 25v$ in (4) gives

$$F(u, v) = u^3 - 81uv^2 + 282v^2 = 1. \qquad \ldots(7)$$

This corresponds to the ring $Z[\theta] = \theta^3 - 81\theta - 282 = 0$, with the fundamental unit

$$\epsilon = 13285773\theta^2 - 68160687\theta - 727721279 = a_1\theta^2 + b_1\theta + c_1.$$

We must find all n such that

$$\epsilon^n = u + v\theta. \qquad \ldots(8)$$

We see that $a_1 \equiv b_1 \equiv O \pmod 3$. Unfortunately $a_1 \equiv O \pmod{3^2}$ and $3 \| b$. Therefore, for prime $p = 3$ we cannot apply Theorem $M1$ or $H3$. However

$$\left|\frac{F(b_1, -a_1)}{(b_1, a_1)^2}\right| = 2 \cdot 3 \cdot 499 \cdot 1429 \cdot 3583 \cdot 49681 \cdot 560837.$$

and for $p = 449$ we find that $n = 448$ is the smallest positive power of ϵ with $a_n \equiv b_n \equiv O \pmod{449}$. Since $a_n \equiv 6286 \pmod{449^2}$, (8) has no solution for $n \neq 0$ by Theorem $H3$. So the only solution of (7) is $U = 1$, $v = 0$ which yields $x = 27$, $y = \pm 141$.

The substitution $a = u_1 - Sv_1$, $b = v_1$ in (2) gives

$$42u_1^3 - 39u_1^2v_1 + 12u_1v_1^2 - v_1^3 = 3.$$

Thus $v_1 \equiv O \pmod 3$. So putting $u_1 = u + v\ v_1 = 3u$ in the above equation we get

$$F(u, v) = 2u^3 + 3uv^2 + 14v^3 = 1. \qquad \ldots(9)$$

Equation (9) defines the ring $Z[1, \alpha, \beta]$ where $\alpha^3 + 6\alpha - 56 = 0$,

$$\beta^3 - 3\beta^2 - 392 = 0 \text{ and } \alpha\beta = 28.$$

We have

$$Z[1, \alpha, \beta] \equiv Z\left[1, \alpha, \frac{6 + \alpha^2}{2}\right] \equiv Z\left[1, \alpha, \frac{\alpha^2}{2}\right].$$

We must check whether $(2, \alpha)$ is a principal ideal in

$$Z\left[1, \alpha, \frac{\alpha^2}{2}\right].$$

Calculations in the field $Q(\alpha)$ yield:

$\alpha^3 + 6\alpha - 56 \equiv (\alpha + 1)^3$ (modulo 3). Since

$$\frac{(\alpha + 1)^2}{3}$$

satisfies $x^3 + 3x^2 - 33x - 147 = 0$, a local basis at 3 is

$$\left[1,\ \alpha,\ \frac{(\alpha+1)^2}{3}\right].$$

Thus

$$\left[1,\ \alpha,\ \frac{4+2\alpha+\alpha^2}{6}\right]$$

is an integral basis for $Q(\alpha)$. The fundamental unit of

$$Q(\alpha) \text{ is } \epsilon_0 = \frac{1885+131\alpha-212\alpha^2}{3}.$$

A module basis for $g(2, \alpha)$ is

$$\left[2,\ \alpha,\ \frac{4+2\alpha+\alpha^2}{3}\right].$$

By Voronoi's method Voronoi[5] we get

$$g = \left(\frac{118+23\theta+7\theta^2}{3}\right).$$

Thus

$$(X) = \left(\frac{2}{\frac{118+23\theta+7\theta^2}{3}}\right) = \left(\frac{-22+10\theta-\theta^2}{6}\right).$$

But this does not belong to the ring

$$Z\left[1,\ \alpha,\ \frac{\alpha^2}{2}\right].$$

Thus (9) is unsoluble.

Finally we put $a = 4u + 5v$, $b = u + v$ in (5), getting

$$F(u, v) = 6u^3 + 9uv^2 + 2v^3 = 1.$$

Let $v_1 = 2v$, $u = u_1$. Then $v_1^3 + 9v_1^2u_1 + 24u_1^3 = 4$ defines the ring $Z[\theta]$ by $\theta^3 - 9\theta^2 - 24 = 0$, with $\epsilon = 2895356395 - 577575819\theta + 28617456\theta^2$. Thus if $\epsilon^n = a_n + b_n\theta + c_n\theta^2$ we have $a_n \equiv 1$, $b_n \equiv 0$, $c_n \equiv O \pmod 2$. An integral basis is

$$\left[1,\ \theta,\ \frac{\theta(\theta+1)}{2}\right].$$

The integer $\alpha = \dfrac{\theta+\theta^2}{2}$ satisfies $\alpha^3 - 45\alpha^2 - 126\alpha - 102 = 0$.

So $\alpha^3 - 45\alpha^2 - 126\alpha - 102 \equiv \alpha^3 + \alpha^2 = \alpha^2(\alpha+1) \pmod 2$.

But $\dfrac{\alpha(\alpha+1)}{2} = \dfrac{132+13\theta+51\theta^2}{4} \notin Z[\theta]$.

Since $2^5 \,\|\, N(132 + 13\theta + 51\theta^2) = 17952$, $2 = \rho_1^2\rho_2$ and there are two non-associated integers of norm 2. One may be taken as

$$\frac{16 - 11\theta + \theta^2}{2}$$

and the other as $269 + 29\theta + 104\theta^2$. Thus there are three nonassociated integers of norn 4, and we get the following equations.

(i) $v_1 + u_1\theta = (-14 - 82\theta + 9\theta^2)(a_n + b_n\theta + c_n\theta^2)$. This implies

$9a_n - b_n - 23c_n = 0$. Therefore $a_n \equiv O \pmod 2$ which is impossible.

(ii) $v_1 + u_1\theta = (2553385 + 275186\theta + 987177\theta^2)(a_n + b_n\theta + c_n\theta^2)$ implies

$a_n + b_n \equiv O \pmod 2$ which is impossible.

(iii) $v_1 + u_1\theta = \left(\dfrac{\theta + \theta + 3\theta^2}{2}\right)(a_n + b_n\theta + c_n\theta^2)$ implies $a_n \equiv O \pmod 2$

which is impossible. Thus (5) has no integral solutions.

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