Indian J. pure appl. Math., 22(4): 313-321, April 1991

LOCALLY BOUNDED TOPOLOGIES ON $F(X)$

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(*Received* 16 *November* 1989; *after revision* 14 *September* 1990; *accepted* 27 *November* 1990)

Let F be an arbitrary field and let $F(X)$ be the field of rational functions over F. Let $p(X)$ be a nonconstant irreducible polynomial in $F[X]$. If $\frac{f(X)}{g(X)}$ is an arbitrary nonzero element of $F(X)$ we can write

$$\frac{f(X)}{g(X)} = p(X)^n \frac{r(X)}{s(X)},$$

where $r(X)$ and $s(X)$ are relatively prime elements of $F[X]$, neither of which is divisible by $p(X)$. We set

$$v_{p(X)}\left(\frac{f(X)}{g(X)}\right) = n, \; v_{p(x)}(0) = \infty$$

and

$$v_{\infty}\left(\frac{f(X)}{g(X)}\right) = \deg g(X) - \deg f(X), \; v_{\infty}(0) = \infty.$$

One can show that $v_{p(X)}$ and v_{∞} are valuations on $F(X)$ which are trivial on F. It is well known that every nontrivial valuation on $F(X)$ which is trivial on F is equivalent to either v_{∞} or $v_{p(X)}$ for some nonconstant irreducible polynomial $p(X)$ in $F[X]$. If T is the supremum of finitely many topologies defined by these valuations, then T is a Hausdorff locally bounded ring topology on $F(X)$ for which F is a bounded set and for which there is a nonzero topological nilpotent element in $F(X)$. Jo-Ann Cohen[2] proved the converse of this, that is, any topology on $F(X)$ having these properties is the supremum of finitely many valuation topologies. In this paper we give a comparatively simple proof of the same by strengthening Theorem 1 of Jo-Ann Cohen.

Let R be a ring and let T be a 'ring topology' on R (that is a topology making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R). A subset S of R is 'bounded' for T if for each neighbourhood U of zero, there exists a neighbourhood V of zero such that $VS \subseteq U$ and $SV \subseteq U$. T is a 'locally bounded topology' on R if there is a bounded neighbourhood of zero. An element c of R is a 'topological nilpotent' if $\lim_{n\to\infty} c^n = 0$. A bounded subfield of a Hausdorff topological ring is discrete (Bourbaki[1] 119, Ex 13).

We recall that a norm $\|\cdot\cdot\|$ on a field K is a function from K to the nonnegative reals satisfying $\|x\| = 0$ if and only if $x = 0$, $\|x - y\| \leq \|x\| + \|y\|$ and $\|xy\| \leq \|x\| \, \|y\|$, for all $x, y \in K$. If $\|\cdot\cdot\|$ is a norm on a field K, for each $\epsilon > 0$, define B_ϵ by $B_\epsilon = \{a \in K / \|a\| < \epsilon\}$. Then $\{B_\epsilon / \epsilon > 0\}$ is a fundamental system of neighbourhoods of zero for a Hausdorff locally bounded ring topology $T_{\|\cdot\|}$ on K. Two norms on K are equivalent if they define the same topology. If the topology defined by $\|\cdot\|$ is nondiscrete then a subset A of K is bounded in norm if and only if A is bounded for the topology defined by the norm. Thus the topology given by a norm on a field is a locally bounded topology. We shall use the following Theorem 6.1 of Cohn[4] : A Hausdorff, locally bounded ring topology on a field K for which there is a nonzero topological nilpotent is defined by a norm.

Let $P = \{p \in F[X] : p$ is an irreducible polynomial$\}$, and let $P' = P \cup \{\infty\}$. For each $p \in P'$, we shall denote by T_p the topology defined by the valuation v_p. Let L be a finite subset of P, then $h = \prod_{p \in L} p$ is a nonzero topological nilpotent for $\sup_{p \in L} T_p$. More generally h/g where $g \in F[X]$ is such that $(g, h) = 1$ is a nonzero topological nilpotent for $\sup_{p \in L} T_p$. We prove the converse of this in Theorem 2. If L is a finite subset of P' containing ∞, then taking $h = \prod_{p \in L \cap P} p$ (the product is taken to be 1 if $L \cap P = \phi$) and $g \in F[X]$ satisfying $(g, h) = 1$ and $\deg g > \deg h$, we get h/g to be a nonzero topological nilpotent for $\sup_{p \in L} T_p$. Further note that, for every topological nilpotent $h/g \in F(X)$ for $\sup_{p \in L} T_p$, $\deg g > \deg h$. Theorem 3 is converse of this.

Theorem 1 — Let F be a field and X a transcendental element over F in some field extension. Let T be a Hausdorff locally bounded ring topology on $F(X)$ for which the subfield F is bounded (and hence discrete) and for which there exists a nonzero topological nilpotent $\alpha_0 = \dfrac{f_0(X)}{g_0(X)} \in F(X)$. Then

1. $R = \{\xi \in F(X) / \xi = 0$ or $\xi = \dfrac{l(X)}{m(X)}$, $(m(X), f_0(X)) = 1$, $\deg l(X) \leq \deg m(X)\}$ is a bounded subring of $F(X)$.

2. There exists a unique monic polynomial $h(X) \in F[X]$ satisfying the following :

   (i) $\dfrac{h(X)}{t(X)}$ is a topological nilpotent for some $t(X) \in F[X]$,

   (ii) if $\dfrac{f(X)}{g(X)}$ is a nonzero topological nilpotent element then $h(X)$ divides $f(X)$ in $F[X]$,

   (iii) either $h(X) = 1$ or $h(X)$ is the product $p_1 \cdot p_2 \cdots p_k$ of finitely many distinct irreducible monic polynomials of $F[X]$.

3. If $h(X) \neq 1$, then $\sup_{1 \leq i \leq k} T_{p_i} \subseteq T$.

Proof : By Cohn's theorem there exists a norm $\|\cdots\|$ defining T.

Then $$\left\| \left(\frac{f_0(X)}{g_0(X)} \right)^n \right\| < 1 \text{ for some } n \geq 1.$$

Let $$\alpha = \frac{f_0^m (X)}{g_0^n (X)} = \frac{f (X)}{g (X)} \quad \text{(say)}.$$

Then α is a topological nilpotent and $\|\alpha\| < 1$.

As F is bounded for T, there exists a positive constant M such that $\|a\| \leq M$ for all $a \in F$.

Let $$\xi = \frac{l (X)}{m (X)} \in R, \ \xi \neq 0.$$

Then ξ satisfies the following conditions:

(i) $(m (X), f (X) = 1$,

(ii) $\deg l (X) \leq \deg m (X) + \deg g (X)$.

Let $f (X) = p_1^{e_1} (X) \, p_2^{e_2} (X) \ldots p_s^{e_s} (X)$, where $e_i \geq 1$ and $s \geq 0$ be the decomposition of $f (X)$ into the product of irreducible polynomials. By the strong approximation theorem (Theorem of Jo-Ann Cohn[3]) there exists an element $\beta_0 \in F (X)$ such that

$v_{p_i} (\xi - \beta_0) \geq e_i$ for $i = 1, 2, \ldots, s$ and $v_p (\beta_0) \geq 0$ for each irreducible polynomial $p \neq p_1, p_2, \ldots, p_s$ in $F [X]$.

The general element $\beta \in F (X)$ satisfying the above properties is given by $\beta = \beta_0 + f (X) \, r (X)$ where $r (X) \in F [X]$. Note that $\beta_0 \in F [X]$ and hence $\beta \in F [X]$. We shall now define $\xi_1, \xi_2, \ldots, \xi_t$ in $F (X)$ and $\beta_1, \beta_2, \ldots, \beta_t$ in $F [X]$ such that

$$\left.\begin{aligned} \xi = \xi_0 &= \alpha \, \xi_1 + \beta_1 \\ \xi_1 &= \alpha \, \xi_2 + \beta_2 \\ \cdots & \quad \cdots \quad \cdots \\ \cdots & \quad \cdots \quad \cdots \\ \xi_{t-1} &= \alpha \, \xi_t + \beta_t \end{aligned}\right\} \qquad \ldots(*)$$

where each ξ_i can be written as $\dfrac{l_i (X)}{m (X)}$ for $i = 1, 2, \ldots, t$ satisfying the following properties :

(i) $(m (X), f (X)) = 1$,

(ii) $\deg l_i (X) \leq \deg m (X) + \deg g (X)$.

Put

$$\xi_1 = \frac{\xi - \beta}{\alpha} = \frac{\xi - \beta_0}{\alpha} - \frac{r (X) \, f (X)}{\alpha} = \frac{\xi - \beta_0}{\alpha} - r (X) \, g (X).$$

One can write $\dfrac{\xi - \beta_0}{\alpha}$ in the form $\dfrac{t (X)}{m (X)}$ for some $t (X) \in F [X]$.

(Note that $(t (X), m (X))$ need not be equal to 1). Therefore

$$\xi_1 = \frac{t (X)}{m (X)} - r (X) \, g (X) = \frac{t (X) - m (X) \, r (X) \, g (X)}{m (X)}.$$

By the division algorithm we choose $r(X) \in F[X]$ such that either $t(X) - m(X) r(X) g(X) = 0$ or its degree is less than the degree of $m(X) g(X)$. For this choice of $r(X) \in F[X]$, ξ_1 can be written as $l_1(X)/m(X)$ satisfying the following:

(i) $(m(X), f(X)) = 1$,

(ii) $\deg l_1(X) \leq \deg m(X) + \deg g(X)$.

Since ξ_1 satisfies the same conditions as ξ, we can repeat the above process to get $\xi_2, \xi_3, \ldots, \xi_t$ in $F(X)$ and $\beta = \beta_1, \beta_2, \ldots, \beta_t$ in $F[X]$ satisfying the desired properties. One can easily check that

$$\deg \beta_i (X) \leq \text{Max} (\deg f(X), \deg g(X) = q \text{ (say)}.$$

Therefore $\quad \beta_i (X) = a_0 + a_1 x + a_2 x^2 + \ldots + a_q x^q$, where $a_i \in F$.

Hence $\quad \|\beta_i\| \leq N'$ where $N' = M(1 + \|X\| + \|X\|^2 + \ldots + \|X\|^q)$.

(Note that N' depends only on α_0.)

Now $\quad m(X)\, \xi_i = l_i (X) \in F[X]$ and

$\deg m(X)\, \xi_i \leq \deg m(X) + \deg g(X)$, therefore there exists a constant N'' depending on ξ and α_0 such that

$$\|m(X)\, \xi_i\| \leq N'' \text{ for all } i = 1, 2, \ldots, t.$$

Therefore $\quad \|\xi_i\| \leq N'''$ where $N''' = N'' \left\| \dfrac{1}{m(X)} \right\|$

which depends only on ξ and α_0. Now eliminating $\xi_1, \xi_2, \ldots, \xi_{t-1}$ from (*) we get

$$\xi = \beta_1 + \alpha\beta_2 + \alpha^2\beta_3 + \ldots + \alpha^{t-1} \beta_t + \alpha^t \xi_t .$$

Therefore

$$\|\xi\| \leq N'(1 + \|\alpha\| + \|\alpha\|^2 + \ldots + \|\alpha\|^{t-1}) + \|\alpha\|^t N'''.$$

Because $\|\alpha\| < 1$, therefore as t tends to ∞ we get $\|\xi\| \leq N$ where

$$N = N' \left(\frac{1}{1 - \|\alpha\|} \right)$$

which depends only on α_0. Therefore R is bounded.

(2) First we prove that if $\alpha = \dfrac{f(X)}{g(X)}$ is a topological nilpotent and if $f(X) = p_1^{r_1}, p_2^{r_2} \ldots p_k^{r_k}$ is its canonical representation as the product of powers of irreducible polynomials of $F[X]$ then there exists $t(X) \in F[X]$ such that $\dfrac{h(X)}{t(X)}$ is a topological nilpotent where $h(X) = p_1, p_2 \ldots p_k$. Put $r = \max \{r_1, r_2, \ldots, r_k\}$. Then $f(X)$ divides $h^r(X)$ in $F[X]$. Take any irreducible polynomial $q(X) \neq p_1, p_2, \ldots, p_k$ and $\xi = \dfrac{h(X)}{q^n(X)}$, where n is a sufficiently large positive integer to be suitably chosen.

Now

$$\xi^r = \frac{h^r(X)}{q^{nr}(X)} = \frac{f(X)\, u(X)}{q^{nr}(X)} = \alpha\beta$$

where $$\beta = \frac{u(X)\, g(X)}{q^{nr}(X)}.$$

Now $\deg u(X)\, g(X) \leq \deg q^{nr}(X)$ for sufficiently large n and $(f(X), q^{nr}(X)) = 1$. The same conditions hold for β^m for all $m \geq 1$. Therefore if N is a constant determined in (1) which depends only on α, then $\|\beta^m\| \leq N$ for all $m \geq 1$ and hence $\|\xi^{rm}\| \leq \|\alpha^m\|\, \|\beta^m\| \leq N\, \|\alpha^m\| < 1$ for large m. Therefore ξ^{rm} and hence ξ is a topological nilpotent.

Now let $h(X)$ be a monic polynomial of smallest degree such that

$$\alpha = \frac{h(X)}{t(X)}$$

is a topological nilpotent for some $t(X) \in F[X]$. Now we prove that if

$$\frac{f(X)}{g(X)}$$

is a topological nilpotent then $h(X)$ divides $f(X)$ in $F[X]$. If $h(X) = 1$ there is nothing to prove. Suppose $h(X) \neq 1$. Therefore it follows from the above that $h(X) = p_1, p_2 \ldots . p_k$ where $p_1, p_2, \ldots, p_k$ are distinct irreducible monic polynomials. Put

$$f(X) = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}\, u(x) \text{ where } e_i \geq 0 \text{ and } (u(X), h(X)) = 1.$$

If N is the constant determined in (1) which depends on

$$\alpha = \frac{h(X)}{t(X)},$$

then $$\left\| \left(\frac{1}{u(X)}\right)^m \right\| \leq N \text{ for all } m \geq 1.$$

Hence $$\left\| \left(\frac{f(X)}{g(X)\, u(X)}\right)^m \right\| \leq N \left\| \left(\frac{f(X)}{g(X)}\right)^m \right\| < 1 \text{ for large } m.$$

Therefore

$$\frac{f(X)}{g(X)\, u(X)} = \frac{p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}}{g(X)}$$

is a topological nilpotent. Therefore by the choice of $h(X)$ together with the result proved in the first paragraph we get that $e_i \geq 1$. Hence $h(X)$ divides $f(X)$. This also proves the uniqueness of $h(X)$.

(3) There exists a topological nilpotent $\alpha = \dfrac{h(X)}{t(X)}$ for some $t(X) \in F[X]$ and $(h(X), t(X)) = 1$. Let K be a given positive integer.

Take

$$\delta = \text{Min}\left\{1, \frac{1}{\|\alpha^{-K}\|}\right\}.$$

Now for every $\xi \in F(X)$ with $\|\xi\| < \delta$, $\|\xi\alpha^{-K}\| < 1$. Therefore $\xi\alpha^{-K}$ is a topological nilpotent of $F(X)$ and hence the numerator of $\xi\alpha^{-K}$ is divisible by $h(X)$. Therefore $v_{p_i}(\xi\alpha^{-K}) \geq 1$ and hence $v_{p_i}(\xi) \geq 1 + K\, v_{p_i}(\alpha) = 1 + K > K$ for all $i = 1, 2,\ldots, k$. Therefore $\sup_{1\leq i\leq k} T_{p_i} \subseteq T$.

This completes the proof of Theorem 1.

We now concentrate on two cases. The first one being when there exists a topological nilpotent element $f(X)/g(X)$ in $F(X)$ with $\deg g(X) \leq \deg f(X)$. This case is taken in Theorem 2. The second one being when for every topological nilpotent element $f(X)/g(X)$ of $F(X)$, $\deg g(X) > \deg f(X)$. This case is taken in Theorem 3. The following Theorem 2 is a generalization of Theorem 1 of Jo-Ann Cohen[2].

Theorem 2 — Let F be a field and X a transcendental element over F in some field extension. Let T be a Hausdorff locally bounded ring topology on $F(X)$ for which the subfield F is bounded (and hence discrete) and for which there exists a nonzero topological nilpotent

$$\alpha_0 = \frac{f_0(X)}{g_0(X)} \in F(X)$$

such that $\deg g_0(X) \leq \deg f_0(X)$. Then

(1) $R = \{\xi \in F(X)/\xi = 0$ or $\xi = l(X)/m(X), (m(X), f_0(X)) = 1\}$ is a bounded subring of $F(X)$ and hence $f_0(X) \in F[X]$ is a nonzero topological nilpotent element. In particular $F[X]$ is bounded.

(2) There exists a unique monic polynomial $h(X) \in F[X]$ such that if $f(X)/g(X)$ is a nonzero topological nilpotent, then $h(X)$ divides $f(X)$. Moreover $h(X) \neq 1$ and $h(X)$ is the product of a sequence $p_1, p_2,\ldots, p_k$ of distinct irreducible monic polynomials of $F[X]$. In fact if we put $J = \{f(X) \in F[X]/f(X)$ is a topological nilpotent of $F(X)\}$, then J is a proper ideal of $F[X]$ and is generated by $h(X)$.

(3) $T = \sup_{1\leq i\leq k} T_{p_i}$.

Proof : (1) As in Theorem 1, there exists a norm $\|\cdot\cdot\|$ defining T. If f_0 is a constant, then $\alpha_0 \neq 0$ is a constant, and a topological nilpotent, contradicting the fact that the topology on F is discrete. Choose a positive integer n such that

$$\|\alpha_0^n\| < \frac{1}{\|f_0(X)\|}.$$

Then $\dfrac{f_0^{n+1}(X)}{g_0^n(X)}$ is a topological nilpotent element with $\left\|\dfrac{f_0^{n+1}(X)}{g_0^n(X)}\right\| < 1$ and moreover if $m(X) \in F[X]$ then $(m(X), f_0(X)) = 1$ if and only if $(m(X), f_0^{n+1}(X)) = 1$. Therefore without loss of generality we can assume that

$$\alpha_0 = \frac{f_0\ (X)}{g_0\ (X)} \text{ satisfies } \|\alpha_0\| < 1 \text{ and } \deg g_0\ (X) < \deg f_0\ (X).$$

Now let N be the constant determined in (1) of Theorem 1 (which depends only on α_0).

Let $$\xi = \frac{l\ (X)}{m\ (X)} \in R.$$

Choose a positive integer t such that

$$\deg l(X) - \deg m(X) \leq t\ [\deg f_0 - \deg g_0].$$

Suppose $f_0\ (X) = p_1^{e_1} \cdot p_2^{e_2} \ldots p_s^{e_s}$ where $e_i \geq 1$, $s > 0$ is a representation of $f_0\ (X)$ as a product of powers of distinct irreducible polynomials. By the weak approximation theorem (Meara[5], page 8, Theorem 11.8), there exists an element $\xi_1 \in F(X)$ satisfying

$$v_\infty \left(\frac{\xi}{\alpha_0^t} - \xi_1\right) \geq t\ (\deg f_0\ (X) - \deg g_0\ (X))$$

and $v_{p_i}\ (\xi_1) \geq 0$ for $i = 1, 2, \ldots, s$.

Now $$v_\infty\ (\xi_1) \geq \text{Min} \left\{ v_\infty \left(\xi_1 - \frac{\xi}{\alpha_0^t}\right), v_\infty \left(\frac{\xi}{\alpha_0^t}\right)\right\} \geq 0,$$

and therefore ξ_1 is of the form

$$\xi_1 = \frac{l_1\ (X)}{m_1\ (X)}$$

with $(m_1\ (X), f_0\ (X)) = 1$ and $\deg l_1\ (X) \leq \deg m_1\ (X)$. Hence $\|\xi_1\| \leq N$. Put $\xi_2 = \xi - \xi_1\ \alpha_0^t$.

Then $$v_\infty\ (\xi_2) = v_\infty \left(\frac{\xi}{\alpha_0^t} - \xi_1\right) + t v_\infty\ (\alpha_0) \geq 0$$

and $v_{p_i}\ (\xi_2) \geq \text{Min}\ \{v_{p_i}\ (\xi), v_{p_i}\ (\xi_1\ \alpha_0^t)\} \geq 0$ for $i = 1, 2, \ldots, s$.

Therefore ξ_2 can be written as

$$\xi_2 = \frac{l_2\ (X)}{m_2\ (X)}$$

where $(m_2\ (X), f_0\ (X)) = 1$ and $\deg l_2\ (X) \leq \deg m_2(X)$. Hence $\|\xi_2\| \leq N$.

Now $$\|\xi\| \leq \|\xi_2\| + \|\xi_1\|\ \|\alpha_0\|^t \leq 2N.$$

Therefore R is bounded. In particular for every polynomial $f(X) \in F\,[X]$, $\|f(X)\| \leq 2N$. Therefore $\|f_0^n\ (X)\| = \|\alpha_0^n\ g_0^n(X)\| \leq \|\alpha_0\|^n\ \|g_0^n\ (X)\| \leq 2N\|\alpha_0\|^n < 1$ for large n and hence $f_0(X)$ is a topological nilpotent.

(2) The existence of $h(X)$ with the required properties follows from (2) of Theorem 1 and that J is a proper ideal of $F[X]$ can be verified as a routine matter because $F[X]$ is bounded. We are to prove $J = \langle h(X) \rangle$. Because there exists a topological nilpotent $h(X)/t(X)$ for some $t(X) \in F[X]$, therefore $h(X)$ is also a

topological nilpotent and hence $<h(X)> \subseteq J$. Conversely if $f(X) \in J$ then because $f(X)$ is topological nilpotent, $h(X)$ divides $f(X)$. Therefore $J \subseteq <h(X)>$ and hence $J = <h(X)>$. From this it also follows that $h(X) \neq 1$. (3) $\sup_{1 \le i \le k} T_{p_i} \subseteq T$ follows from (3) of Theorem 1. Since $h(X)$ is a topological nilpotent, there exists a positive real number K such that for every $\xi \in F(X)$ with $\xi = l(X)/m(X)$, $(m(X), h(X)) = 1$, $\|\xi\| \le K$ by (1). Now let $\epsilon > 0$ be given. There exists a positive integer m such that $\|(h(X))^{m+1}\| < \epsilon/K$. If $\xi \in F(X)$ with $v_{p_i}(\xi) > m$ for $i = 1, 2, \ldots, k$; then ξ can be written as

$$\xi = \frac{h(X)^{m+1} f(X)}{g(X)}$$

where $(g(X), h(X)) = 1.$

Therefore $\|\xi\| \le \|h(X)^{m+1}\| \left\| \frac{f(X)}{g(X)} \right\| < \frac{\epsilon}{K} K = \epsilon.$

Therefore $T \subseteq \sup_{1 \le i \le k} T_{p_i}.$

Theorem 3 — Let F be a field and X a transcendental element over F in some field extension. Let T be a Hausdorff locally bounded ring topology on $F(X)$ for which the subfield F is bounded (and hence discrete) and for which there exists a nonzero topological nilpotent in $F(X)$. If every topological nilpotent $f(X)/g(X)$ of $F(X)$ satisfies $\deg g(X) > \deg f(X)$, then

(1) There exists a unique monic polynomial $h(X) \in F[X]$ such that if $f(X)/g(X)$ is a topological nilpotent element then $h(X)$ dividends $f(X)$ and if $h(X) \neq 1$, $h(X)$ is the product of a sequence $p_1, p_2, \ldots, p_k$ of distinct irreducible monic polynomials of $F[X]$.

(2) (i) If $h(X) = 1$, $T = T_\infty$,
(ii) if $h(X) \neq 1$, $T = \sup\{T_\infty, \sup_{1 \le i \le k} T_{p_i}\}$.

Proof : (1) follows from (2) of Theorem 1.

(2) There exists a topological nilpotent α of the form $h(X)/t(X)$ for some $t(X) \in F[X]$. By assumption $\deg t(X) > \deg h(X)$.

Let M be a given positive integer. Take

$$\delta = \text{Min}\left\{1, \frac{1}{\|\alpha^{-M}\|}\right\}.$$

Now for every $\xi \in F(X)$ with $\|\xi\| < \delta$, $\|\xi\alpha^{-M}\| < 1$. Therefore $\xi\alpha^{-M}$ is a topological nilpotent element of $F(X)$ and hence by assumption $v_\infty(\xi\alpha^{-M}) \ge 1$, Therefore $v_\infty(\xi) \ge 1 + M v_\infty(\alpha) > M$. Hence $T_\infty \subseteq T$. We now prove the converse.

(i) In this case $\alpha = 1/t(X)$. Let $\epsilon > 0$ be given. Let N be a positive constant depending on α as determined in (1) of Theorem 1. Choose a positive integer n such that $\|\alpha^n\| < \epsilon/N$. Take $M = n \deg t(X)$. (Not that $\deg t(X) \ge 1$ by assumption.) Now for any $\xi \in F(X)$ with $v_\infty(\xi) > M$, $\xi\alpha^{-n}$ will be of the form $l(X)/m(X)$ with $\deg l(X) \le \deg m(X)$ and therefore $\|\xi\alpha^{-n}\| \le N$. Now $\|\xi\| \le \|\xi\alpha^{-n}\| \|\alpha^n\| < \epsilon$. This shows that $T \subseteq T_\infty$. Therefore $T = T_\infty$.

(ii) In this case $\alpha = h(X)/t(X)$, where $h(X)$ is the product of a sequence p_1, $p_2,\ldots,p_k$ of distinct irreducible monic polynomials of $F[X]$.
By (3) of Theorem 1, $\sup\limits_{1\le i\le k} T_{p_i} \subseteq T$, therefore

$\sup\{T_\infty, \sup T_{p_i}\} \subseteq T$. Let $\epsilon > 0$ be given and let N be a constant depending only on $\alpha = h(X)/t(X)$ as determined in (1) of Theorem 1. Choose a positive integer n such that $\|\alpha^n\| < \epsilon/N$. Take $M = n\,[\deg t(X)\text{-}\deg h(X)] \geq n$. Now for any $\xi \in F(X)$ with $v_\infty(\xi) > M$ and $v_{p_i}(\xi) > M$ for $i = 1, 2,\ldots,k$; $\xi\,\alpha^{-n}$ will be of the form $l(X)/m(X)$ with $(m(X), h(X)) = 1$ and $\deg l(X) \leq \deg m(X)$. Therefore $\|\xi\alpha^{-n}\| \leq N$ and hence $\|\xi\| \leq \|\xi\alpha^{-n}\|\,\|\alpha^n\| < \epsilon$. This shows that $T \subseteq \sup\{T_\infty, \sup\limits_{1\le i\le k} T_{p_i}\}$. Therefore

$$T = \sup\{T_\infty, \sup_{i\le i\le k} T_{p_i}\}.$$

ACKNOWLEDGEMENT

The authors are thankful to the referees for their valuable suggestions.

REFERENCES

1. N. Bourbaki, *Topologie Generale*, Ch 3-4, Hermann, Paris, 1964.
2. Jo-Ann Cohen, *Pacific J. Math.* **70** (1977), 125-32.
3. Jo-Ann Cohen, *Pacific J. Math.* **87** (1980), 59-63.
4. P. M. Cohn *Proc. Cambridge Phil. Soc.* **50** (1954), 159-77.
5. O. T. O. Meara *Introduction to Quadratic Forms*, Springer-Verlg, New York, 1971.