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STRONGLY NONLINEAR VARIATIONAL INEQUALITY PROBLEM AND ITS ALTERNATIVE FORM AND COMPLEMENTARITY PROBLEM

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This paper introduces an alternative form and complementarity problem of strongly nonlinear variational inequality problem. Equivalence of nonlinear variational inequality problem and its alternative form and complementarity problem are established. Also existence of solutions of strongly nonlinear variational inequality problem and its complementarity problem are established.

1. INTRODUCTION

In this paper we discuss the strongly nonlinear variational inequality problem (SNVIP) which was originally studied by Noor[5]. In section 2 we introduce an alternative form of SNVIP and prove its equivalence with SNVIP under certain suitable conditions. Section 3 deals with strongly nonlinear complementarity problem (SNCP) and its equivalence with SNVIP. Finally in section 4 we establish existence theorems for SNVIP and SNCP.

Let K be a nonempty closed convex subset of a real Hilbert space H. Let T and A be nonlinear operators from K into H. Then the strongly nonlinear variational inequality problem (SNVIP) is to find $u \in K$ such that

$$(Tu, v - u) \geq (Au, v - u) \text{ for all } v \in K. \qquad \ldots(1)$$

This type of problem is considered and studied by Noor[5]. If $A = 0$, then the above problem reduces to the variational inequality requiring to find $u \in K$ such that

$$(Tu, v - u) \geq 0 \text{ for all } v \in K$$

and this was studied by Hartman and Stampacchia[1] and many others.

2. STRONGLY NONLINEAR VARIATIONAL INEQUALITY PROBLEM AND ITS ALTERNATIVE FORM

In this section we introduce an alternative form of SNVIP and prove its equivalence with SNVIP under certain suitable conditions. Let S denote the set of all solutions of SNVIP given by (1), that is

$$S = \{u \in K : (Tu, v - u) \geq (Au, v - u) \text{ for all } v \in K\}.$$

The alternative form which we introduce now is to find $u \in K$ such that

$$(Tv, v - u) \geq (Av, v - u) \text{ for all } v \in K \quad \ldots(2)$$

and let V denote the set of all solutions $u \in K$ of (2).

We now quote some definitions which will be needed in the sequel. A mapping $T : H \to H$ is said to be monotone if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in H$ and strictly monotone if strict inequality holds for $x \neq y$. T is strongly monotone if there exists a constant $c > 0$ such that

$$(Tx - Ty, x - y) \geq c \|x - y\|^2 \text{ for all } x, y \in H.$$

T is said to be hemicontinuous on $K \subset H$ if for all $x, y \in K$, the map

$$t \mapsto T(ty + (1 - t) x)$$

of $[0, 1]$ to H is continuous in the usual topology of $[0, 1]$ and weak topology of H.

Theorem 1 – Let T and A be monotone and hemicontinuous maps from a convex set K in a Hilbert space H into H. Then $S = V$.

Proof : Let $u \in V$. Let v be an arbitrary element of K. Since K is convex, $v_t = (1 - t) u + tv \in K$ for $0 < t < 1$. Hence we have

$$(Tv_t, v_t - u) \geq (Av_t, v_t - u)$$

Since T and A are hemicontinuous, letting $t \to 0$ we get

$$(Tu, v - u) = (Au, v - u)$$

and hence $u \in S$.

Conversely, let $u \in S$. Since T and A are monotone, for any $v \in K$,

$$(Tv, v - u) \geq (Tu, v - u)$$

and

$$(Av, v - u) \geq (Au, v - u).$$

By substracting the second inequality from the first one we get

$$(Tv - Av, v - u) \geq (Tu - Au, v - u)$$
$$\geq 0.$$

Thus $(Tv, v - u) \geq (Av, v - u)$ and hence $u \in V$.
This completes the proof.

3. Equivalence of Strongly Nonlinear Variational Inequality Problem and its Complementarity Problem

In this section we introduce strongly nonlinear complementarity problem (SNCP) and establish its equivalence with SNVIP.

Recall that, for a nonempty closed convex cone K in a real Hilbert space H, K^* is defined by

$$K^* = \{u \in H : (Tu, v) \geq (Au, v) \text{ for all } v \in K\}.$$

Strongly nonlinear complementarity problem (SNCP) is to find $u \in H$ such that

$$u \in K, Tu \in K^*, (Tu, u) = (Au, u).$$

Let C denote the set of all solutions of SNCP. Then we have

Theorem 2 – If K is a closed convex cone in a Hilbert space H, then $C = S$.

PROOF : Let $u \in S$. Then since $0 \in K$ we have

$$(Tu, u) \leq (Au, u).$$

Similarly by taking $v = 2u \in K$, we get

$$(Tu, u) \geq (Au, u).$$

Thus $(Tu, u) = (Au, u)$. It remains to show that $Tu \in K^*$. Assume, to the contrary, that $Tu \notin K^*$. Then there exists some $v_0 \in K$ such that

$$\begin{aligned}(Au, v_0) &> (Tu, v_0)\\ &= (Tu, u) + (Tu, v_0 - u)\\ &= (Au, u) + (Tu, v_0 - u)\\ &\geq (Au, u) + (Au, v_0 - u)\\ &= (Au, v_0).\end{aligned}$$

Thus $(Au, v_0) > (Au, v_0)$ and this is impossible. Hence the result follows.

Conversely, let $u \in C$. Then for any $v \in K$,

$$\begin{aligned}(Tu, v - u) &= (Tu, v) - (Tu, u)\\ &= (Tu, v) - (Au, u)\\ &\geq (Au, v) - (Au, u)\\ &= (Au, v - u).\end{aligned}$$

Hence $u \in S$ and this completes the proof.

4. Extence of Solutions of Strongly Nonlinear Vartiational Inequality Problem and its Complementarity Problem

In this final section we shall prove the existence theorems for solutions of strongly nonlinear variational inequality problem and strongly nonlinear complementarity problem in $\mathbb{R}^n$.

Theorem 3 – Let T and A be continuous maps from a nonempty compact convex set $K \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Then there exists a point $u_0 \in K$ such that

$$(Tu_0, v - u_0) \geq (Au_0, v - u_0)$$

for all $v \in K$.

PROOF : For a fixed $x_0 \in K$ consider the map $h : K \to \mathbb{R}$ defined by

$$h(w) = \|w - x_0 + Tx_0 - Ax_0\|.$$

Clearly h is continuous on K. Since K is compact, h attains a minimum, say at w_0. It is also easy to verify that w_0 is unique. Observe that the correspondence $x_0 \mapsto w_0$

defines a continuous function of K into itself. Thus by Brouwer's theorem we have a fixed point u_0 and this is the required point of the theorem.

Theorem 4 – Let T and A be monotone and continuous maps from a nonempty closed convex cone $K \subset \mathbb{R}^n$ into $\mathbb{R}^n$ such that there exists a $u_0 \in K$ with $Tu_0 \in$ int K^*. Then there exists $u \in \mathbb{R}^n$ such that

$$u \in K,\ Tu \in K^*,\ (Tu,\ u) = (Au,\ u).$$

Proof : For any $e \in$ int K^* and $r > 0$ write

$$D_r(e) = \{u \in K : 0 \leq (e,\ u) \leq r\},$$

$$D_r^0(e) = \{u \in K : 0 < (e,\ u) < r\},$$

$$S_r(e) = \{u \in K : (e,\ u) = r\}.$$

For any $e \in$ int K^* and $r > 0$, $D_r\ (e)$ is compact and convex (see Lemma 2.2 of Nanda and Nanda[4] for the proof). Therefore it follows from Theorem 3 that for each $r > 0$, there is an $u_r \in D_r\ (e)$ such that for all $v \in D_T\ (e)$,

$$(Tu_r,\ v - u_r) \geq (Au_r,\ v - u_r). \qquad \ldots(3)$$

Since $0 \in D_r\ (e)$, it follows that

$$(Tu_r,\ u_r) \leq (Au_r,\ u_r).$$

If there exists $e \in$ int K^* and $r > 0$ such that $u_r \in D_r^0\ (e)$, then there exists $\lambda > 1$ such that $\lambda u_r \in S_r\ (e) \subset D_r\ (e)$. Hence from (3) it follows that

$$\begin{aligned}(Tu_r - Au_r,\ u_r) &\leq (Tu_r - Au_r,\ \lambda u_r)\\ &= \lambda(Tu_r - Au_r,\ u_r).\end{aligned}$$

Since $(Tu_r,\ u_r) \leq (Au_r,\ u_r)$, the above inequality is impossible unless

$$(Tu_r,\ u_r) = (Au_r,\ u_r).$$

To complete the proof assume, if possible, $u_r \in S_r\ (e)$ for all $e \in$ int K^* and all $r > 0$. By the hypothesis there exists $u_0 \in K$ with $Tu_0 \in$ int K^*. Set $e = Tu_0$. Choose

$$r > (Tu_0,\ u_0) > (Au_0,\ u_0)$$

Now $u_0 \in D_r^0\ (Tu_0)$. Since T and A are monotone,

$$(Tv - Av,\ v - u_0) \geq (Tu_0 - Au_0,\ v - u_0) > 0$$

for all $v \in S_r\ (Tu_0)$. But $u_r \in S_r\ (Tu_0)$. Hence

$$(Tu_r - Au_r,\ u_r - u_0) > 0$$

or $$(Tu_r,\ u_r - u_0) > (Au_r,\ u_r - u_0).$$

Since $u_0 \in D_r^0\ (Tu_0) \subset D_r\ (Tu_0)$, from (3) we have

$$(Tu_r,\ u_r - u_0) \leq (Au_r,\ u_r - u_0).$$

This contradiction shows that the assumption "that $u_r \in S_r\ (e)$ for all $r > 0$ and $e \in$ int K^*" is not true for $e = Tu_0$. Thus the proof of the theorem is reduced to the previous case, the case where there exists $c \in$ int K^* and $r > 0$ such that $u_r \in D_r^0\ (e)$ and this completes the proof.

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