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SOME INEQUALITIES IN HYPERGEOMETRIC FUNCTIONS USING STATISTICAL TECHNIQUES

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The positivity property of mathematical expectation is used to derive several inequalities concerning hypergeometric functions ${}_2F_1$ and ${}_3F_2$. Some interesting special cases are pointed out.

1. INTRODUCTION

It appears that there are very few inequalities on hypergeometric functions in the literature. Askey[1] points out four and suggests more work in that direction. The inequalities of this paper come from statistical considerations and are based on the positivity of certain mathematical expectations.

The results in section 2 are directly from Koti[2], and we outline his work at the end of this section. Sections 2 and 3 contain inequalities in Gauss' hypergeometric functions followed by some interesting special cases. In section 4 inequalities in ${}_2F_1$ and ${}_3F_2$ are derived using a result due to Mathai and Saxena[4].

The following notation is used throughout this paper. For details reference is made to Mathai and Saxena[3] and Rainville[5]. Let Re(*) denote the real part of (*). For $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$, let $\Gamma(\alpha)$ and $B(\alpha, \beta)$ denote, respectively, the gamma and beta functions. Let

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt, \quad 0 < x < 1;$$

$$\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0 \quad \text{...(1.1)}$$

and

$$I_x(\alpha, \beta) = B_x(\alpha, \beta)/B(\alpha, \beta). \quad \text{...(1.2)}$$

A generalized hypergeometric function ${}_pF_q$, is defined by

$${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!} \quad \text{...(1.3)}$$

in which no denominator β_j is allowed to be zero or a negative integer, and for example, $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$, $(\alpha)_0 = 1$. In this paper we deal with ${}_2F_1$ and ${}_3F_2$ functions only.

By $X \sim G\,(a, b)$ we mean that the random variable X has the gamma density

$$f_X(x) = (\Gamma(a)\, b^a)^{-1}\, e^{-x/b}\, x^{a-1},\ x > 0,\ a > 0,\ b > 0.$$

Let X and Y be statistically independently distributed such that $X \sim G\,(\alpha, 1)$ and $Y \sim G\,(\beta, 1)$. Consider the non-negative random variable defined by

$$Z = \begin{cases} rX - Y & \text{if } rX > Y, \\ 0 & \text{otherwise,} \end{cases}$$

where r is a positive real number. Then by the positivity property of mathematical expectation in this case, it follows that, for a positive integer k,

$$E\,(Z^k) = c \int_0^r (r - t)^k \frac{t^{\beta-1}}{(l + t)^{\alpha+\beta+k}}\, dt \geq 0 \qquad \ldots(1.4)$$

where $c = \Gamma(\alpha + p + k)/(\Gamma(\beta))$. See also Koti[2].

Furthermore, using (1.4) Koti[2] proved that, for a real positive number r and a positive integer k,

$$\sum_{i=0}^{k} \binom{k}{i} r^i\, (-1)^{k-i}\, B(\alpha + i,\ \beta + k - i)\, [1 - I_{1/(1+r)}\,(\alpha + i, \beta + k - i)] \geq 0. \quad \ldots(1.5)$$

This gives rise to the following result:

(i) For an even positive integer k,

$$\sum_{j=0}^{k/2} \binom{k}{2j} r^{2j}\, B(\alpha + 2j,\ \beta + k - 2j)\, [1 - I_{1/(1+r)}\,(\alpha + 2j, \beta + k - 2j)]$$
$$\geq \sum_{j=0}^{k/2-1} \binom{k}{2j+1} r^{2j+1}\, B(\alpha + 2j + 1,\ \beta + k - 2j - 1)\, [1 - I_{1/(1+r)}\,(\alpha + 2j + 1, \beta + k - 2j - 1)].$$
$$\ldots(1.6)$$

(ii) For an odd positive integer k,

$$\sum_{j=0}^{[k/2]} \binom{k}{2j+1} r^{2j+1}\, B(\alpha + 2j + 1,\ \beta + k - 2j - 1)\, [1 - I_{1/(1+r)}\,(\alpha + 2j + 1, \beta + k - 2j - 1)].$$
$$\geq \sum_{j=0}^{[k/2]} \binom{k}{2j} r^{2j}\, B(\alpha + 2j,\ \beta + k - 2j)\, [1 - I_{1/(1+r)}\,(\alpha + 2j, \beta + k - 2j)]. \ldots(1.7)$$

where $[k/2]$ is the largest integer less than or equal to $k/2$.

When $K = 1$, the inequality in (1.7) simplifies to yield

$$r\alpha \{1 - I_{1/(1+r)} (\alpha + 1, \beta)\} \geq \beta \{1 - I_{1/(1+r)} (\alpha, \beta + 1)\}.$$

2. Inequalities in ${}_2F_1$ Functions

We recall that for $0 \leq x \leq 1$ and $\alpha, \beta > 0$,

$$I_x (\alpha, \beta) = 1 - I_{1-x} (\beta, \alpha) \qquad ...(2.1)$$

and that $B(\alpha, \beta) = B(\beta, \alpha)$. By using (2.1), (1.5) can be written as

$$\sum_{i=0}^{k} \binom{k}{i} r^i (-1)^{k-i} \int_0^{r/(1+r)} u^{\beta+k-i-1} (1-u)^{\alpha+i-1} du \geq 0. \qquad ...(2.2)$$

An incomplete beta function and a hypergeometric function ${}_2F_1$, are related by

$$\int_0^x u^{\alpha-1} (1-u)^{\beta-1} du = \frac{x^\alpha}{\alpha} {}_2F_1 (\alpha, 1-\beta; 1+\alpha; x) \qquad ...(2.3)$$

for all $0 < x \leq 1$. Combining (2.2) and (2.3), we obtain

$$x^{\beta+k} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \left(\frac{1}{1-x}\right)^i \frac{1}{\beta+k-i}$$
$$\times {}_2F_1 (\beta+k-i, 1-\alpha-i; 1+\beta+k-i; x) \geq 0. \qquad ...(2.4)$$

Dividing throughout by $x^{\beta+k}$ establishes the following result.

Theorem 2.1 — Let $0 < x < 1$ and $\alpha, \beta > 0$.

(i) For an even positive integer k,

$$\sum_{j=0}^{k/2} \binom{k}{2j} \left(\frac{1}{1-x}\right)^{2j} \frac{1}{\beta+k-2j}$$
$${}_2F_1 (\beta+k-2j, 1-\alpha-2j; 1+\beta+k-2j; x)$$
$$\geq \sum_{j=0}^{k/2-1} \binom{k}{2j+1} \left(\frac{1}{1-x}\right)^{2j+1} \frac{1}{\beta+k-2j-1}$$
$${}_2F_1 (\beta+k-2j-1, -\alpha-2j; \beta+k-2j; x).$$

(ii) For an odd positive integer k,

$$\sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left(\frac{1}{1-x}\right)^{2j+1} \frac{1}{\beta+k-2j-1}$$
$${}_2F_1 (\beta+k+2j-1, -\alpha-2j; \beta+k-2j; x)$$
$$\geq \sum_{j=0}^{[k/2]} \binom{k}{2j} \left(\frac{1}{1-x}\right)^{2j} \frac{1}{\beta+k-2j}$$
$${}_2F_1 (\beta+k-2j, 1-\alpha-2j; 1+\beta+k-2j; x)$$

where $[k/2]$ is the largest integer less than or equal to $k/2$.

Corollary 2. 1 – As a special case for $k = 1$, we have

$$(1 + \beta)\, {}_2F_1\,(\beta, -\alpha; 1 + \beta; x) \geq (1 - x)\, \beta\, {}_2F_1\,(1 + \beta, 1 - \alpha; 2 + \beta; x).$$

3. Inequalities in ${}_2F_1$ – Series

Note that the integral in (1.4) remains non-negative if $(r - t)^k$ is replaced by any arbitrary function $g(t)$ such that $g(t) \geq 0$ for $0 < t < r$, where r is a positive real number. Let

$$g(t) = \begin{cases} (1 + t/r)^{\lambda} & \text{for } 0 < t < r, \\ 0 & \text{otherwise} \end{cases}$$

where λ is a nonzero real number. We have

$$c \int_0^r \left(1 + \frac{t}{r}\right)^{\lambda} \frac{t^{\beta-1}}{(1 + t)^{\alpha+\beta+k}}\, dt \geq 0$$

where c (> 0) is defined in (1.4). Because $(1 + t/r)^{\lambda} > 1$ for all $0 < t < r$, $\lambda > 0$, we note that

$$\int_0^r \left(1 + \frac{t}{r}\right)^{\lambda} \frac{t^{\beta-1}}{(1 + t)^{\alpha+\beta+k}}\, dt \geq \int_0^r \frac{t^{\beta-1}}{(1 + t)^{\alpha+\beta+k}}\, dt > 0. \qquad \ldots(3.1)$$

Expending $\left(1 + \dfrac{t}{r}\right)^{\lambda}$ as a binomial series, (3.1) yields

$$\sum_{j=0}^{\infty} \frac{(-\lambda)_j}{j!} (-1)^j \int_0^r \left(\frac{t}{r}\right)^j \frac{t^{\beta-1}}{(1 + t)^{\alpha+\beta+k}}\, dt \geq \int_0^r \frac{t^{\beta-1}}{(1 + t)^{\alpha+\beta+k}}\, dt.$$

Substituting $w = t/(1 + t)$, we get

$$\sum_{j=0}^{\infty} \frac{(-\lambda)_j}{j!} (-1)^j r^{-j} B_{r/(1+r)}\,(\beta + j, \alpha + k - j) \geq B_{r/(1+r)}\,(\beta, \alpha + k). \qquad \ldots(3.2)$$

Now applying (2.3) and replacing $r/(1+r)$ by x, we rewrite (3.2) as

$$\sum_{j=1}^{\infty} \frac{(-\lambda)_j}{j!} (-j)^j \frac{(1 - x)^j}{\beta + j}\, {}_2F_1\,(\beta + j, 1 - \alpha + j; 1 + \beta \cdot + j; x) \geq 0. \qquad \ldots(3.3)$$

Note that for $\lambda <, 0$ (3.3) can be modified as

$$\sum_{j=0}^{\infty} \frac{(-\lambda)_j}{j!} (-1)^j \frac{(1 - x)^j}{\beta + j}\, {}_2F_1\,(\beta + j, 1 - \alpha + j; 1 + \beta + j; x) \geq 0. \qquad \ldots(3.4)$$

Thus, we have the following theorem.

Theorem 3.1 – (a) For $\lambda > 0$ and $0 < x < 1$,

$$\sum_{j=1}^{\infty} \frac{(-\lambda)_{2j}}{(2j)!} \frac{(1-x)^{2j}}{\beta+2j} \, {}_2F_1(\beta+2j, 1-\alpha-k+2j; 1+\beta+2j; x)$$

$$\geq \sum_{j=0}^{\infty} \frac{(-\lambda)_{2j+1}}{(2j+1)!} \frac{(1-x)^{2j+1}}{\beta+2j+1} \, {}_2F_1(\beta+2j+1, -\alpha-k+2j; 2+\beta+2j; x). \quad \ldots(3.5)$$

(b) For $\lambda < 0$ and $0 < x < 1$,

$$\sum_{j=0}^{\infty} \frac{(-\lambda)_{2j}}{(2j)!} \frac{(1-x)^{2j}}{\beta+2j} \, {}_2F_1(\beta+2j, 1-\alpha-k+2j; 1+\beta+2j; x)$$

$$\geq \sum_{j=C}^{\infty} \frac{(-\lambda)_{2j}}{(2j+1)!} \frac{(1-x)^{2j+1}}{\beta+2j+1} \, {}_2F_1(\beta+2j+1, -\alpha-k+2j; 2+\beta+2j; x). \quad \ldots(3.6)$$

Substituting $\lambda = \frac{1}{2}$ in (3.5), we have the following corollary.

Corollary 3.1 – For $0 < x < 1$,

$$\frac{1}{2} \frac{1-x}{1+\beta} \, {}_2F_1(\beta+1, 2-\alpha-k; 2+\beta; x)$$

$$+ \frac{1}{16} \frac{(1-x)^3}{3+\beta} \, {}_2F_1(\beta+3, 4-\alpha-k; 4+\beta; x) + \ldots$$

$$\geq \frac{1}{8} \frac{(1-x)^2}{2+\beta} \, {}_2F_1(\beta+2, 3-\alpha-k; 3+\beta; x)$$

$$+ \frac{5}{128} \frac{(1-x)^4}{4+\beta} \, {}_2F_1(\beta+4, 5-\alpha-k; 5+\beta; x) + \ldots$$

Letting $\lambda = -\frac{1}{2}$ in (3.6), we have the following corollary.

Corollary 3.2 – For $0 < x < 1$,

$$\frac{1}{\beta} \, {}_2F_1(\beta, 1-\alpha-k; 1+\beta; x)$$

$$+ \frac{3}{8} \frac{(1-x)}{2+\beta} \, {}_2F_1(\beta+2, 3-\alpha-k; 3+\beta; x)$$

(*Equation continued on page* 394)

$$+ \frac{35}{128} \frac{(1-x)^4}{4+\beta} {}_2F_1(\beta+4, 5-\alpha-k; 5+\beta; x) + \ldots$$

$$\geq \frac{1}{2} \frac{(1-x)}{1+\beta} {}_2F_1(\beta+1, 2-\alpha-k; 2+\beta; x)$$

$$+ \frac{5}{16} \frac{(1^3-x)}{3+\beta} {}_2F_1(\beta+3, 4-\alpha-k; 4+\beta; x)$$

$$+ \frac{63}{256} \frac{(1-x)^5}{5+\beta} {}_2F_1(\beta+5, 6-\alpha-k; 6+\beta; x) + \ldots .$$

Finally, substituting $\lambda = -1$ in (3.6) we have the following corollary.

Corollary 3.3 – For $0 < x < 1$,

$$\frac{1}{\beta} {}_2F_1(\beta, 1-\alpha-k; 1+\beta; x)$$

$$+ \frac{(1-x)^2}{\beta+2} {}_2F_1(\beta+2, 3-\alpha-k; 3+\beta; x)$$

$$+ \frac{(1-x)^4}{\beta+4} {}_2F_1(\beta+4, 5-\alpha-k; \beta+5; x) + \ldots$$

$$\geqq \frac{1-x}{\beta+1} {}_2F_1(\beta+1, 2-\alpha-k; 1+\beta; x)$$

$$+ \frac{(1-x)^3}{\beta+3} {}_2F_1(\beta+3, 4-\alpha-k; 4+\beta; x)$$

$$+ \frac{(1-x)^5}{\beta+5} {}_2F_1(\beta+5, 6-\alpha-k; \beta+6; x) + \ldots$$

4. Inequalities in ${}_2F_1$ and ${}_3F_2$ Functions

Let X_1 and X_2 bc stochastically independent real beta variates with density functions

$$f_j(x_j) = \frac{1}{B(\alpha_j, \beta_j)} x_j^{\alpha_j} (1-x_j)^{\beta_j - 1}, \; 0 < x_j < 1,$$

$$\alpha_j > 0, \; \beta_j > 0, \; j = 1, 2. \qquad \ldots(4.1)$$

Methai and Saxena[4] showed that the p.d.f., g_1 of $U = X_1 X_2$ is

$$g_i(u) = c\, u^{\alpha_2 - 1} (1-u)^{\beta_1 + \beta_2 - 1} {}_2F_1(\alpha_2 + \beta_2 - \alpha_1, \beta_1; \beta_1 + \beta_2; 1-u)\, I_{(0,1)}(u)$$

where $I_A(\cdot)$ is the indicator function of A and

$$c = B(\beta_1, \beta_2) / \sum_{l=1}^{2} B(\alpha_l, \beta_l).$$

Therefore, the p.d.f., g_1 of $V = 1 - U$, is

$$g_1(v) = c\, v^{\beta_1+\beta_2-1} (1 - v)^{\alpha_2-1} {}_2F_1 (\alpha_2 + \beta_2 - \alpha_1, \beta_1; \beta_1 + \beta_2; v)_{(0,1)} (v).$$

Consider the random variable

$$Z = \begin{cases} t - V & \text{if } V \leqq t, \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(4.2)$$

where t is a positive constant $\leqq 1$. Now for a real positive interger λ,

$$E[Z^{\lambda-1}] = c \int_0^t v^{\beta_1+\beta_2-1} (1 - v)^{\alpha_2-1} (t - v)^{\lambda-1} \times {}_2F_1 (\alpha_2 + \beta_2 - \alpha_1, \beta_1; \beta_1 + \beta_2; v)\, dv \geqq 0. \qquad \ldots(4.3)$$

If α_2 is an integer, from (4.3), we have

$$\sum_{j=0}^{\alpha_2} \binom{\alpha_2}{j} (-1)^{\alpha_2-j} \int_0^t v^{\beta_1+\beta_2+\alpha_2-j-1} (t - v)^{\lambda-1} \times {}_2F_1 (\alpha_2 + \beta_2 - \alpha_1, \beta_1; \beta_1 + \beta_2; v)\, dv \geqq 0. \qquad \ldots(4.4)$$

By using Theorem 38 in Rainville[5], p. 104, we can rewrite (4.4) as

$$\sum_{j=0}^{\alpha_2} \binom{\alpha_2}{j} (-1)^{\alpha_2-j} B(\beta_1 + \beta_2 + \alpha_2 - j, \lambda)\, t^{\beta_1+\beta_2+\alpha_2+\lambda-j-1} \times {}_3F_2 (\alpha_2 + \beta_2 - \alpha_1, \beta_1, \beta_1 + \beta_2 + \alpha_2 - j; \beta_1 + \beta_2, \beta_1 + \beta_2 + \alpha_2 + \lambda - j; t) \geqq 0.$$

Dividing throughout by $\Gamma(\lambda)\, t^{\beta_1+\beta_2+\alpha_2+\lambda}$ and by using Theorem 9 in Rainville[5], p. 23, we have proved the following theorem.

Theorem 4.1 – Let $0 < t \leqq 1$.

$$\sum_{j=0}^{\alpha_2} \binom{\alpha_2}{j} (-1)^{\alpha_2-j} \frac{t^{-j-1}}{(\beta_1+\beta_2+\alpha_2 - j)_\lambda} {}_3F_2 (\alpha_2 + \beta_2 - \alpha_1, \beta_1, \beta_1 + \beta_2 + \alpha_2 - j; \beta_1 + \beta_2, \beta_1 + \beta_2 + \alpha_2 + \lambda - j; t) \geqq 0. \qquad \ldots(4.5)$$

Separating the even and odd terms in (4.5), one can list results analogous to the ones in Theorem 2.1 and special cases can be considered for various positive integer values of α_2 and λ.

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