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INTEGRALS INVOLVING GENERALIZED LAURICELLA'S FUNCTION AND THE *H*-FUNCTION OF SEVERAL COMPLEX VARIABLES

V. B. L. Chaurasia and Sanjeev Tyagi

Department of Mathematics, University of Rajasthan
Jaipur 302 004

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This paper presents some finite integrals involving generalized Lauricella function and the *H*-function of several complex variables. The special cases of our integrals yield interesting results. As generalized Lauricella function and the *H*-function of several complex variables are of general nature, the results, on specializing the parameters, lead to many known and unkonwn results.

1. Introduction and the Main Result

For the *H*-function of several complex variables defined by Srivastava and Panda[6], we derive the following integral transformation:

$$\int_0^1 x^{\epsilon-1}(1-x)^{\beta} F_{\sigma: B'; \ldots; B^{(s)}}^{\nu: A'; \ldots; A^{(s)}}\left(\begin{matrix}[(\alpha_\nu): \theta', \ldots, \theta^{(s)}]\\ [(\beta_\sigma): \psi', \ldots, \psi^{(s)}]\end{matrix}; \begin{matrix}0, 0\\ 1, 1\end{matrix}; \begin{matrix}\lambda, \lambda]: [(a'): \phi']; \ldots; [(a^{(s)}): \phi^{(s)}]; ----; ----;\\ \mu, \mu]: [(b'): \delta']; \ldots; [(b^{(s)}): \delta^{(s)}]; [\alpha+1: 1]; [\beta: 1: 1];\end{matrix}$$

$$z_1', \ldots, z_r, \ldots, z_s', -xt, (1-x)t\Big)$$

$$\times H(z_1 x^{h_1}, \ldots, z_r x^{h_r})\, dx$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\nu} (\alpha_j)_{n\lambda_j}}{(\alpha+1)_n (\beta+1)_n \prod_{j=1}^{\sigma} (\beta_j)_{n\mu_j}}$$

$$F_{\sigma: B'; \ldots; B^{(s)}}^{\nu: A'; \ldots; A^{(s)}}\left(\begin{matrix}[(\alpha_\nu+n\lambda_\nu): \theta', \ldots, \theta^{(s)}]: [(a'): \phi']; \ldots; [(a^{(s)}): \phi^{(s)}];\\ [(\beta_\sigma+n\mu_\sigma): \psi', \ldots, \psi^{(s)}]: [(b'): \delta']; \ldots; [(b^{(s)}): \delta^{(s)}];\end{matrix} z_1', \ldots, z_s\right)$$

$$\frac{(-t)^n \Gamma(\beta+n+1)}{n!} H_{A+2, D+2:}^{0, T+2:} \begin{matrix}(u', v'); \ldots; (u^{(r)}, v^{(r)})\\ (M', N'); \ldots; (M^{(r)}, N^{(r)})\end{matrix}$$

$$\left(\begin{matrix}[1-\epsilon: h_1, \ldots, h_r], [1+\epsilon+\alpha: h_1, \ldots, h_r], [(d): \eta', \ldots, \eta^{(r)}]:\\ [(c): \xi', \ldots, \xi^{(r)}], [1-\epsilon+\alpha+n: h_1, \ldots, h_r], [-\beta-\epsilon-n: h_1, \ldots, h_r]:\\ [(m'): \rho']; \ldots; [(m^{(r)}): \rho^{(r)}];\\ [(l'): \tau']; \ldots; [(l^{(r)}): \tau^{(r)}];\end{matrix} z_1, \ldots, z_r\right) \qquad \ldots(1.1)$$

where $\operatorname{Re}(\beta) > -1$, $\operatorname{Re}\left(\epsilon + \sum_{i=1}^{r} h_i\, l_j^{(i)}/\tau f^{(i)}\right) > 0$,

$j = 1, \ldots, u^{(i)}$; $h_i > 0$, $T_i > 0$, $|\arg(z_i)| < \frac{1}{2} T_i \pi$,

$i = 1, \ldots, r$; $|t| < 1$ and the series on the right is convergent.

2. Proof

To prove (1.1), we start with the following result [see Srivastava and Daoust[5], eqn. (1.2), p. 15]

$$F_{\sigma:\ B';\ \ldots;\ B^{(s)};\ 1;\ 1;\ [(\beta_\sigma):\psi', \ldots, \psi^{(s)}, \mu, \mu]:[(b'):\delta'];\ \ldots;}^{\nu:\ A';\ \ldots;\ A^{(s)};\ 0;\ 0\ [(\alpha_\nu):\theta', \ldots, \theta^{(s)}, \lambda, \lambda]:[(a'):\phi'];\ \ldots;}$$

$$[(a^{(s)}):\phi^{(s)}];\ ----;\ ----;\ [(b^{(s)}):\delta^{(s)}];\ [\alpha+1:1];\ [\beta+1:1];\ z_1', \ldots, z_s - xt, (1-x)\,t\Big)$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\nu} (\alpha_j)_{n\lambda_j}}{(a+1)_n\,(\beta+1)_n \prod_{j=1}^{\sigma} (\beta_j)_{n\mu_j}} P_n^{(\alpha,\beta)}(1-2x)$$

$$F_{\sigma:\ B';\ \ldots;\ B^{(s)}}^{\nu:\ A';\ \ldots;\ A^{(s)}} \begin{matrix}[(\alpha_\nu + n\lambda_\nu):\theta', \ldots, \theta^{(s)}]:[(a'):\phi'];\ \ldots;\ [(a^{(s)}):\phi^{(s)}];\\ [(\beta_\sigma + n\mu_\sigma):\psi', \ldots, \psi^{(s)}]:[(b'):\delta'];\ \ldots;\ [(b^{(s)}):\delta^{(s)}];\end{matrix} z_1', \ldots, z_s\Big)\, t^n$$

where λ_i, $\theta_i^{(j)}$, $i = 1,\ldots,\sigma$

and $j = 1,\ldots,s$; μ_i, $\psi_i^{(j)}$, $i = 1,\ldots,\nu$

and $j = 1,\ldots,s$; ϕ_i, $\delta_k^{(j)}$, $i = 1,\ldots, A^{(j)}$, $j = 1,\ldots,s$

and $k = 1,\ldots, B^{(j)}$

are all real and positive, (α_ν) is taken to abbreviate the sequence of parameters $\alpha_1,\ldots,\alpha_\nu$, $(a^{(j)})$ stands for the sequence of $A^{(j)}$ parameters

$$a_1^{(j)},\ldots, a_{A^{(j)}}^{(j)},\ j = 1,\ldots,s,$$

with similar interpretations for (β_σ) and $(b^{(j)})$, $j = 1,\ldots,s$ and $F(z_1',\ldots,z_s')$ denotes the generalized Lauricella function of s complex variables $z_1',\ldots,z_s'$ (see Srivastava and Daoust[4], p. 454).

Now multiplying both sides of (2.1) by $x^{\epsilon-1}(1-x)^{\beta}$. $H(z_1 x^{h_1},\ldots,z_r x^{h_r})$ and integrating it with respect to x from 0 to 1. Evaluating the right hand side thus obtained, by interchanging the order of integration and summation (which is justified due to the absolute convergence of the integral involved in the process) and then interpreting the inner integral with the help of a known result [see Srivastava and Panda[6], eqn. (2.2), p. 131], we arrive at the desired result.

3. Particular Cases

(i) On specializing the parameters in a suitable manner and letting $\alpha_1 = \gamma$, $\alpha_2 = \delta$, $\lambda_1 = \lambda_2 = 1$ in (1.1), we obtain the following integral

$$\int_0^1 x^{\epsilon-1}(1-x)^{\beta} H(z_1 x^{h_1},\ldots,z_r x^{h_r}) \times F_c^{(s+2)}\left[\gamma, \delta ; \omega_1,\ldots,\omega_s, \alpha+1, \beta+1 ; z_1',\ldots,z_s', -xt, (1-x)t\right] dx$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n (\delta)_n (-t)^n \Gamma(\beta+n+1)}{(\alpha+1)_n (\beta+1)_n n!} \times F_c^{(s)}[\gamma+n, \delta+n; \omega_1,\ldots,\omega_s ; z_1',\ldots,z_s'] \times H^{0,T+2:(u',v');\ \ldots\ ;(u^{(r)},v^{(r)})}_{A+2,D+2:(M',N');\ \ldots\ ;(M^{(r)},N^{(r)})}$$

$$\left(\begin{matrix}[1-\epsilon : h_1, \ldots, h_r], [1-\epsilon+\alpha : h_1, \ldots, h_r], [(d) : \eta', \ldots, \eta^{(r)}] : \\ [(c) : \xi', \ldots, \xi^{(r)}], [1-\epsilon+\alpha+n : h_1, \ldots, h_r], [-\beta-\epsilon-n : h_1, \ldots, h_r] : \\ [(m') : \rho'] ; \ldots ; [(m^{(r)}) : \rho^{(r)}] ; \\ [(l') : \tau'] ; \ldots ; [(l^{(r)}) : \tau^{(r)}] ; \end{matrix} z_1,\ldots,z_r\right) \qquad \ldots(3.1)$$

valid under the same conditions obtainable from 61.1).

(ii) For $r = 2$, the result in (3.1) reduces to a known result obtained by Chaurasia[1] (eqn. (1.8.1), p. 35).

(iii) Putting $z_1' = \cdots = z_s' = 0$ in (1.1), we obtain the following integral involving generalized Kampé de Fériet function (cf. Srivastava and Daoust[3], p. 199; see also Srivastava and Daoust[4], p. 450).

$$\int_0^1 x^{\epsilon-1}(1-x)^{\beta} H(z_1 x^{h_1},\ldots, z_r x^{h_1}) \times S_{\sigma:1;1}^{\nu:0;0}\left(\begin{matrix}[(\alpha_\nu) : \lambda, \lambda] : - - - - ; - - - - ; \\ [(\beta_\sigma) : \mu, \mu] : [\alpha+1 : 1]; [\beta+1 : 1] ;\end{matrix} -xt, (1-x)t\right) dx$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\nu} \Gamma(\alpha_j+n\lambda_j)(-t)^n \Gamma(\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\prod_{j=1}^{\sigma}\Gamma(\beta_j+n\mu_j) n!} \times H^{0,T+2:(u',v');\ \ldots\ ;(u^{(r)},v^{(r)})}_{A+2,D+2:(M',N');\ \ldots\ ;(M^{(r)},N^{(r)})}$$

$$\left(\begin{matrix}[1-\epsilon : h_1, \ldots, h_r], [1-\epsilon+\alpha : h_1, \ldots, h_r], [(d) : \eta', \ldots, \eta^{(r)}] : \\ [(c) : \xi', \ldots, \xi^{(r)}], [1-\epsilon+\alpha+n : h_1, \ldots, h_r], [-\beta-\epsilon-n : h_1, \ldots, h_r] : \\ [(m') : \rho'] ; \ldots ; [(m^{(r)}) : \rho^{(r)}] ; \\ [(l') : \tau'] ; \ldots ; [(l^{(r)}) : \tau^{(r)}] ; \end{matrix} z_1,\ldots,z_r\right) \qquad \ldots(3.2)$$

valid under the samer conditions obtainable from (1.1).

(iv) If in eqn. (3.2), we replace ν by $\nu + 2$ and then take $\nu = \sigma = 2$, also let $\alpha_1 = \gamma$, $\alpha_2 = \delta$, $\alpha_3 = \alpha + 1 = \beta$, $\alpha_4 = \beta + 1 = \beta_2$ and choose $\lambda_i = 1 = \mu_j$ where $i = 1,\ldots,\nu + 2$ and $j = 1,\ldots, \sigma$, then it reduces to a known result obtained by G. S. Olkha and V. B. L. Chaurasia[2] (eqn. (2.1), p. 311).

(v) By taking $T = A = D = 0$ in (1.1), we derive an interesting result involving the product of r mutually independent H-functions

$$\int_0^1 x^{\epsilon-1}(1-x)^{\beta} F_{\sigma:\, B';\ \ldots;\ B^{(s)};\ 1;\ 1}^{\nu:\, A';\ \ldots;\ A^{(s)};\ 0;\ 0}\left(\begin{matrix}[(\alpha_\nu):\theta',\ \ldots,\ \theta^{(s)}],\\ [(\beta_\sigma):\psi',\ \ldots,\psi^{(s)},\end{matrix}\right.$$

$$\left.\begin{matrix}\lambda,\ \lambda]:[(a'):\phi'];\ \ldots;\ [(a^{(s)}):\phi^{(s)}];\ ----;\ ----;\\ \mu,\ \mu]:[(b'):\delta'];\ \ldots;\ [(b^{(s)}):\ \delta^{(s)}];\ [\alpha+1:1];\ [\beta+1:1];\end{matrix}\right.$$

$$z_1',\ldots,\ z_s,\ldots,\ z_s',\ -xt,\ (1-x)\ t\Big) \times \sum_{i=1}^{r} H_{M^{(i)},\ N^{(i)}}^{u^{(i)}v^{(i)}}\left[z_i\, x^{hi}\right] dx$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\nu}(\alpha_j)_{n\lambda_j}}{(\alpha+1)_n\ (\beta+1)_n \prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}} \cdot \frac{(-t)^n\,\Gamma(\beta+n+1)}{n!}$$

$$\times F_{\sigma:\, B';\ \ldots;\ B^{(s)}}^{\nu:\, A';\ \ldots;\ A^{(s)}}\left(\begin{matrix}[(\alpha_\nu+n\lambda_\nu):\theta',\ \ldots,\ \theta^{(s)}]\ [(a'):\phi'];\ \ldots;\ [(a^{(s)}):\phi^{(s)}];\\ [(\beta_\sigma+n\mu_\sigma):\psi',\ \ldots,\ \psi^{(s)}]\ [(b'):\ \delta'];\ \ldots;\ [(b^{(s)}):\delta^{(s)}];\end{matrix}\right. z_1',\ldots,\ z_s\Big)$$

$$H_{2,\ 2:\ (M',N');\ \ldots;\ (M^{(r)},\ N^{(r)})}^{0,\ 2:(u',\ v');\ \ldots;\ (u^{(r)},\ v^{(r)})}\left(\begin{matrix}[1-\epsilon:h_1,\ \ldots,h_r],\ [1-\epsilon+\alpha:h_1,\ \ldots,\ h_r],\ [(d):\eta',\ \ldots,\ \eta^{(r)}]:\\ [(c):\xi',\ \ldots,\ \xi^{(r)}],\ [1-\epsilon+\alpha+n:h_1,\ \ldots,\ h_r],\ [-\beta-\epsilon-n:h_1,\ldots,\ h_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(m'):\rho'];\ \ldots;\ [(m^{(r)}):\rho^{(r)}];\\ [(l'):\tau'];\ \ldots;\ [(l^{(r)}):\tau^{(r)}];\end{matrix}\right. z_1,\ldots,z_r\Big) \qquad \ldots(3.3)$$

valid under the same conditions obtained in (1.1).

(vi) Taking $r = 1$ in (3.3), we get the following result

$$\int_0^1 x^{\epsilon-1}(1-x)^{\beta} F_{\sigma:\, B';\ \ldots;\ B^{(s)};\ 1;\ 1}^{\nu:\, A';\ \ldots;\ A^{(s)};\ 0;\ 0}\left(\begin{matrix}[(\alpha_\nu):\theta',\ \ldots,\ \theta^{(s)},\ \lambda,\ \lambda]:\\ [(\beta_\sigma):\psi',\ \ldots,\psi^{(s)},\mu,\ \mu]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(a'):\phi'];\ \ldots;\ [(a^{(s)}):\phi^{(s)}];\ ----;\ ----;\\ [(b'):\delta'];\ \ldots;\ [(b^{(s)}):\ \delta^{(s)}];\ [\alpha+1:1];\ [\beta+1:1];\end{matrix}\right. z_1',\ldots,\ z_\sigma',\ -xt,\ (1-x)\ t\Big)$$

$$\times\ H_{M',N'}^{u',v'}\left[z_1\, x^{h_1}\right] dx$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\nu}(\alpha_j)_{n\lambda_j}}{(\alpha+1)_n\ (\beta+1)_n \prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}}$$

$$F_{\sigma: B'; \ldots; B^{(s)}}^{\nu: A'; \ldots; A^{(s)}}\left(\begin{matrix}[(\alpha_\nu + n\lambda_\nu) : \theta', \ldots, \theta^{(s)}] : [(a') : \phi']; \ldots; [(a^{(s)}) : \phi^{(s)}]; \\ [(\beta_\sigma + n\mu_\sigma) : \psi', \ldots, \psi^{(s)}] : [(b') : \delta']; \ldots; [(b^{(s)}) : \delta^{(s)}];\end{matrix} z_1', \ldots, z_s\right)$$

$$\times H_{M'+2, N'+2}^{u', v'+2}\left[z_1 \,\middle|\, \begin{matrix}(1-\epsilon : h_1), (1-\epsilon+\alpha : h_1), (m_{M'}', \rho_{M'}') \\ (l_{N'}', \tau_{N'}'), (1-\epsilon+\alpha+n : h_1), (-\epsilon-\beta-n : h)\end{matrix}\right] \quad \ldots(3.4)$$

where Re $(\beta) > -1$, $h_1 > 0$,

Re $(\epsilon + h_1 l_j'/\tau_j') > 0$, $j = 1, \ldots, u'$;

$T_1 > 0$, $|t| < 1$, $|\arg(z_1)| < \dfrac{T_1\pi}{2}$ and the series on the right is convergent.

Here $\left(T_1 \equiv \sum_{i=1}^{v'} \rho_i' - \sum_{i=v'+1}^{M'} \rho_i' + \sum_{i=1}^{u'} \tau_i' - \sum_{i=u'+1}^{N'} \tau_i'\right)$.

(vii) On specializing the parameters in a suitable manner and letting $\alpha_1 = \gamma$, $\alpha_2 = \delta$, $\lambda_1 = \lambda_2 = 1$ in (3.4), we get a known integral obtained by Chaurasia[1] (eqn. (1.8.4), p. 38).

On specializing the parameters, generalized Lauricella's function and the H-function of several complex variables may be transformed into G-functions, E-functions, Lauricella's functions, Appell's functions, Kampé de Fériet functions, Bessel functions, and several other higher transcendental functions in one or more arguments. Therefore, the integral transformations for various other functions of one or more variables can be obtained as special cases of our result.

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