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ON INVARIANT SUBMANIFOLDS OF CONTACT METRIC MANIFOLDS

HIROSHI ENDO

Kohnodai Senior High School, 2-4-1, Kohnodai, Ichikawa-shi Chiba-ken, 272, Japan

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A characterization of the invariant submanifold in a contact metric manifold of $\bar{\phi}$-sectional curvature is studied. Furthermore, we will consider an invariant Einstein submanifold under the same condition.

INTRODUCTION

Kon[7] has proved the following two results.

Theorem A – Let $\bar{\mathbf{M}}^{2r+1}$ be a normal contact metric manifold of constant $\bar{\phi}$-sectional curvature $\bar{\mathbf{K}}$ and M^{2n+1} be an invariant submanifold of $\bar{\mathbf{M}}^{2r+1}$. Then M^{2n+1} is totally geodesic if and only if $S_c \geqq n^2 (\bar{\mathbf{K}} + 3) + n (\bar{\mathbf{K}} + 1)$.

Theorem B – If there is an invariant Einstein submanifold M^{2n+1} in a normal contact metric manifold $\bar{\mathbf{M}}^{2r+1}$ of constant $\bar{\phi}$-sectional curvature $\bar{\mathbf{K}}$, then $\bar{\mathbf{K}} \geqq 1$. When and only when $\bar{\mathbf{K}} = 1$, M^{2n+1} is totally geodesic.

The purpose of this paper is to generalize Theorem *A* in the case where $\bar{\mathbf{M}}^{2r+1}$ is a contact metric manifold and to generalize Theorem *B* in the case where $\bar{\mathbf{M}}^{2r+1}$ is a *K*-contact Riemannian manifold.

1. PRELIMINARIES

Let $\bar{\mathbf{M}}^{2r+1}$ be a $(2r+1)$-dimensional contact metric manifold with the structure tensors $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$. Then they satisfy

$$\left.\begin{aligned} &\bar{\phi}\bar{\xi} = 0,\ \bar{\eta}(\bar{\xi}) = 1,\ \bar{\phi}^2 = -I + \bar{\eta} \otimes \bar{\xi},\ \bar{\eta}(\bar{X}) = \bar{g}(\bar{\xi}, \bar{X}) \\ &\bar{\eta} \circ \bar{\phi} = 0,\ \bar{g}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \bar{\eta}(\bar{X})\,\bar{\eta}(\bar{Y}), \\ &\bar{g}(\bar{X}, \bar{\phi}\bar{Y}) = d\bar{\eta}(\bar{X}, \bar{Y}) \end{aligned}\right\} \quad \ldots(1.1)$$

for any vector fields $\bar{\mathbf{X}}$ and $\bar{\mathbf{Y}}$ on $\bar{\mathbf{M}}^{2r+1}$. Now we define an operator $\bar{\mathbf{h}}$ by $\bar{\mathbf{h}} = \frac{1}{2}\mathcal{L}\bar{\phi}$, where $\mathcal{L}$ denotes Lie differentiation. Then $\bar{\mathbf{h}}$ is symmetric and the vector field $\bar{\xi}$ is Killing if and only if $\bar{\mathbf{h}}$ vanishes[1,2]. If $\bar{\xi}$ is a Killing vector field, contact metric manifold $\bar{\mathbf{M}}^{2r+1}$ is called a *K*-contact Riemannian manifold. Moreover if we have

$$(\bar{\nabla}_{\bar{\mathbf{X}}}\bar{\phi})\,\bar{\mathbf{Y}} = -\bar{\eta}(\bar{\mathbf{Y}})\bar{\mathbf{X}} + \bar{\mathbf{g}}(\bar{\mathbf{X}}, \bar{\mathbf{Y}})\bar{\xi},\ \bar{\mathbf{M}}^{2r+1}$$

is called a normal contact metric manifold (Sasakian manifold), where $\bar{\nabla}$ denotes the covariant differentiation with respect to $\bar{\mathbf{g}}$. In the sequel, a contact metric manifold will be denoted simply by $\bar{\mathbf{M}}^{2r+1}$.

Let M^{2n+1} be a $(2n+1)$-dimensional submanifold of $\bar{\mathbf{M}}^{2r+1}$. By N_A $(A = 1,2,\ldots,2(r-n))$ we denote local mutually orthogonal unit vector fields normal to M^{2n+1}. Let ∇ be the Riemannian connection on M^{2n+1} determined by the induced metric g. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_A h_A(X, Y) N_A,$$

$$\bar{\nabla}_X N_A = -H_A X + \sum_B L_{BA}(X) N_B \qquad \ldots(1.2)$$

where h_A and H_A are the second fundamental forms, and L_{BA} is the third fundamental form of M^{2n+1}. The forms h_A and H_A satisfy $h_A(X, Y) = g(H_A X, Y) = g(X, H_A Y) = h_A(Y, X)$. And for any vector fields X, Y, Z and W on M^{2n+1}, the Gauss equation is given by

$$\bar{\mathbf{g}}(\bar{\mathbf{R}}(X,Y)Z,W) = g(R(X,Y)Z,W) - \sum_B g(H_B Y,Z)\, g(H_B X,W) + \sum_B g(H_B X,Z)\, g(H_B Y,W) \qquad \ldots(1.3)$$

where $\bar{\mathbf{R}}$ is the Riemannian curvature tensor of $\bar{\mathbf{M}}^{2r+1}$ and R is the Riemannian curvature tensor of M^{2n+1}.

M^{2n+1} is said to be invariant if $\bar{\phi}X$ is tangent to M^{2n+1} for any tangent vector field X on M^{2n+1} and $\bar{\xi}$ is tangent to M^{2n+1}. Any invariant submanifold M^{2n+1} of a contact metric manifold with the induced structure (ϕ, ξ, η, g) is also a contact metric manifold (see Yano and Kon[10]).

If $\bar{\mathbf{M}}^{2r+1}$ is a K-contact Riemannian manifold, an invariant submanifold M^{2n+1} is a K-contact Riemannian manifold concerning the induced structure (ϕ, ξ, η, g). If $\bar{\mathbf{M}}^{2r+1}$ is a normal contact metric manifold, in the same way, M^{2n+1} is a normal contact metric manifold (see Yano and Kon[10]). If we define $h = \frac{1}{2}\mathcal{L} \wedge \phi$ for an invariant submanifold of a contact metric manifold $\bar{\mathbf{M}}^{2r+1}$, then we have the result that h is symmetric and the vector field ξ is Killing if and only if h vanishes. If a unit vector $\bar{\mathbf{X}} \in T_m(\bar{\mathbf{M}}^{2r+1})$ (where $T_m(\bar{\mathbf{M}}^{2r+1})$ denotes tangent space at a point m of $\bar{\mathbf{M}}^{2r+1}$) is orthogonal to $\bar{\xi}$, then the sectional curvature $\bar{\mathbf{K}}(\bar{\mathbf{X}}, \bar{\phi}\bar{\mathbf{X}}) = \bar{\mathbf{g}}(\bar{\mathbf{R}}(\bar{\mathbf{X}}, \bar{\phi}\bar{\mathbf{X}})\bar{\phi}\bar{\mathbf{X}}, \bar{\mathbf{X}})$ is called the $\bar{\phi}$-sectional curvature of the plane $\{\bar{\mathbf{X}}, \bar{\phi}\bar{\mathbf{X}}\}$. If $\bar{\mathbf{K}}(\bar{\mathbf{X}}, \bar{\phi}\bar{\mathbf{X}})$ does not depend on the choice of $\bar{\mathbf{X}} \in T_m(\bar{\mathbf{M}}^{2r+1})$, $\bar{\mathbf{X}} \perp \bar{\xi}$, we denote it by $\bar{\mathbf{K}}_m$.

Define

$$s^* = \sum_{i,j=0}^{2n} g(R(\tilde{E}_i, \tilde{E}_j)\, \phi\tilde{E}_j, \phi\tilde{E}_i)$$

where $\{\tilde{E}_i\}$ is an orthonormal basis. We need, in the sequel, the following theorem and lemmas.

Theorem 1.1[3,4] – Any invariant submanifold M^{2n+1} of a contact metric manifold $\bar{\mathbf{M}}^{2r+1}$ is minimal.

Lemma 1.1[9] – For any contact metric manifold M^{2n+1}, we have $\|\nabla\phi\|^2 - 4n \geqq 0$ and $s^* - S + 4n^2 = \text{Tr } h^2 + \frac{1}{2}\{\|\nabla\phi\|^2 - 4n\} \geqq 0$. Moreover M^{2n+1} is a normal contact metric manifold if and only if $\|\nabla\phi\|^2 = 4n$ or equivalently $s^* - S + 4n^2 = 0$, where S denotes the scalar curvature of M^{2n+1} and Tr h^2 denotes the trace of h^2.

Lemma 1.2[2] – Let M^{2n+1} be an invariant submanifold of a contact metric manifold $\bar{\mathbf{M}}^{2r+1}$. Then $g(Q\xi,\xi) = 2n - \text{Tr } h^2$, where Q denotes the Ricci operator of M^{2n+1}.

Lemma 1.3[5] – If M^{2n+1} is an invariant submanifold of a contact metric manifold $\bar{\mathbf{M}}^{2r+1}$, then $H_A\xi = 0$.

Lemma 1.4[6] – Let M^{2n+1} be an invariant submanifold of a contact metric manifold $\bar{\mathbf{M}}^{2r+1}$. Then Tr $(\phi H_B)^2 = \text{Tr } H_B^2$.

2. Main Results

Theorem 2.1 – Let $\bar{\mathbf{M}}^{2r+1}$ be a contact metric manifold of pointwise constant ϕ-sectional curvature $\bar{\mathbf{K}}$ and M^{2n+1} be an invariant submanifold of $\bar{\mathbf{M}}^{2n+1}$. Then the scalar curvature S of M^{2n+1} satisfies the inequality

$$S \leqq n^2 (\bar{\mathbf{K}} + 3) + n (\bar{\mathbf{K}} + 1).$$

Equality holds if and only if M^{2n+1} is totally geodesic and Sasakian.

Proof : In virtue of our assumption, we have

$$\bar{\mathbf{g}} (\bar{\mathbf{R}} (X, \phi X) X, \phi X) + \bar{\mathbf{K}}_m \|X\|^4 = 0$$

at any point $m \in M^{2n+1}$ and for any vector $X \in T_m (M^{2n+1})$, $X \perp \xi$. It is clear that this condition implies

$$\bar{\mathbf{g}} (\bar{\mathbf{R}} (\phi X, \phi^2 X) \phi X, \phi^2 X) + \bar{\mathbf{K}}_m \|\phi X\|^4 = 0 \qquad \ldots(2.1)$$

at any point $m \in M^{2n+1}$ and for any vector $X \in T_m (M^{2n+1})$. Set

$$\bar{\mathbf{P}} (X,Y,Z,W) = \bar{\mathbf{g}} (\bar{\mathbf{R}} (\phi X, \phi^2 Y) \phi Z, \phi^2 W) + \bar{\mathbf{K}}_m g(\phi X, \phi Z) g(\phi Y, \phi W).$$

Then tensor $\bar{\mathbf{P}}$ satisfies $\bar{\mathbf{P}} (X,Y,Z,W) = \bar{\mathbf{P}} (Z,W,X,Y)$. Therefore (2.1) is equivalent to

$$\begin{aligned} &\bar{\mathbf{P}} (X,Y,Z,W) + \bar{\mathbf{P}} (X,Y,W,Z) + \bar{\mathbf{P}} (Y,X,Z,W) \\ &+ \bar{\mathbf{P}} (Y,X,W,Z) + \bar{\mathbf{P}} (X,W,Y,Z) + \bar{\mathbf{P}} (X,W,Z,Y) \\ &+ \bar{\mathbf{P}} (W,X,Y,Z) + \bar{\mathbf{P}} (W,X,Z,Y) + \bar{\mathbf{P}} (X,Z,Y,W) \\ &+ \bar{\mathbf{P}} (X,Z,W,Y) + \bar{\mathbf{P}} (Z,X,Y,W) + \bar{\mathbf{P}} (Z,X,W,Y) = 0 \qquad \ldots(2.2) \end{aligned}$$

for arbitrary $X, Y, Z, W \in T_m(M^{2n+1})$. Choosing a ϕ-basis

$$\{E_0 = \xi, E_1, \ldots E_n, E_{n+1} = \phi E_1, \ldots, E_{2n} = \phi E_n\}$$

in $T_m(M^{2n+1})$, taking $X = W = E_i$, $Y = Z = E_j$ into (2.2) and summing over i and j we obtain

$$\sum_{i,j=0}^{2n} \{\bar{\mathbf{P}}(E_i, E_j, E_j, E_i) + \bar{\mathbf{P}}(E_i, E_j, E_i, E_j) + \bar{\mathbf{P}}(E_i, E_i, E_j, E_j)\} = 0.$$

Hence, using the definition of $\bar{\mathbf{P}}$, the Gauss equation (1.3), the first Bianchi identity, Theorem 1.1 and Lemmas 1.2, 1.3, and 1.4, we find, after some lengthy computation,

$$4n(n+1)\bar{\mathbf{K}} - 3s^* - S + 2(2n - \operatorname{Tr} h^2) = 4 \sum_B \operatorname{Tr}(H_B^2).$$

Hence, using Lemma 1.1, we get

$$n(n+1)\bar{\mathbf{K}} + 3n^2 + n - S = \frac{5}{4} \operatorname{Tr} h^2 + \frac{3}{8}\{\|\nabla\phi\|^2 - 4n\} + \sum_B \operatorname{Tr} H_B^2 \qquad \ldots(2.3)$$

which completes the proof.

Theorem 2.2 — If there is an invariant Einstein submanifold M^{2n+1} in a K-contact Riemannian manifold $\bar{M}^{2r+1}$ of pointwise constant $\bar{\phi}$-sectional curvature $\bar{\mathbf{K}}$, then $\bar{\mathbf{K}} \geqq 1$. Equality holds if and only if M^{2n+1} is totally geodesic and Sasakian.

Proof : Since M^{2n+1} is an Einstein manifold, we have $S = 2n(2n+1)$ by means of Lemma 1.2. Then (2.3) leads to

$$n(n+1)(\bar{\mathbf{K}} - 1) = \frac{3}{8}\{\|\nabla\phi\|^2 - 4n\} + 4 \sum_B \operatorname{Tr} H_B^2$$

which completes the proof of Theorem 2.2.

Now we shall give an example of a non-Sasakian contact metric manifold of constant ϕ-sectional curvature. Let E^4 be the 4-dimensional Euclidean space with rectangular coordinates X^k ($k = 1, 2, 3, 4$). We regard E^4 as a flat Kaehlerian manifold. Let M^3 be the hypersurface defined by the equation

$$(X^3)^2 + (X^4)^2 = \left(\frac{1}{2}\right)^2.$$

This is a product manifold of 1-dimensional sphere S^1 with 2-dimensional Euclidean space E^2. Thus, M^3 is flat, i.e., its curvature R vanishes identically. Moreover, M^3 is a contact hypersurface and carries a contact metric structure (see p. 79 and Theorem 3.1 of Okumura[8]). Since all sectional curvatures of plane sections containing ξ are equal to 1 on a Sasakian manifold, M^3 is a non-Sasakian contact metric manifold with a constant ϕ-sectional curvature 0.

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