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ELEMENTARY SUBDIVISIONS AND INFINITE GRAPHS

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In this note, several families of infinite graphs and all finite graphs are proved to be reconstructible from the collection of their elementary subdivisions.

We use the terminology in Bondy[1] for terms not defined here. Accordingly, a graph admits loops and multiple edges. A finite graph has finite number of vertices and edges. An infinite graph is one with a countably infinite number of vertices and a countable number of edges. If $x = uv$ is an edge of a graph G, then x is subdivided when it is replaced by a new vertex w and the edges uw and wv. The new graph thus obtained is called an elementary subdivision of G and is denoted by G_x. We abbreviate an elementary subdivision as e.s. and elementary subdivisions as e.ss. Each edge of G gives an e.s. of G. All the e.ss. of G have the same degree sequence and the degree sequence of G can be obtained by dropping an entry 2 from that of any e.s. of G. Let w be a vertex of degree 2 adjacent to the vertices u and v. The process of deleting w from G and adding the edge uv is called suppression of the vertex w.

If there exists a bijection f between the edge sets of twe graphs G and H such that $G_x \cong H_{f(x)}$ for every edge x of G, then G and H are called subdivision-isomorphic. G is said to be subdivisions-reconstructible if every graph subdivision-isomorphic to G is isomorphic to G. G is said to be subsidivion-set-reconstructible if every graph having the same set of e.ss as G is isomorphic to G. Subdivision-set-reconstructible graphs are obviously subdivision-reconstructible.

An edge joining vertices of degrees m and n is said to have degree pair (m, n) or (n, m). A (finite) stretch in a graph is either (i) a maximal finite path of length at least 2 in which all internal vertices have degree 2, or (ii) a cycle of length at least 2 in which at most one vertex has degrees greater than 2. An infinite stretch is a one way infinite path in which the degree of the first vertex is not 2 and the degrees of all other vertices are 2.

We first prove an elementary result.

Result 1 – A finite graph with at least one edge is subdivision-set-reconstructible.

PROOF : Take an e.s. G_1 that has the longest stretch among all e.ss. This longest stretch is unique in G_1 and suppressing a vertex of this stretch we get G.

If we are given a collection (repetition of members allowed) of finite graphs, we can determine whether this is the collection of e.ss of a graph or not as follows:

If the degree sequences of all the graphs in the given collection are not the same, or if the stretch of maximum length occurring in the given collection is not unique in some e.s. containing such a stretch, the collection has not come from a graph. Otherwise, choose an e.s. with a stretch of maximum length and suppress a vertex in that stretch. If the resulting graph has the given collection as the collection of e.ss., the decision is obvious. Otherwise, there does not exist a graph having the given collection as its e.ss.

Even though establishing the subdivision-reconstructibility of finite graphs is not difficult, that for infinite graphs is not easy. However, the following theorems show that infinite graphs in several families are subdivision-reconstructible.

Theorem 1 – An infinite graph G is subdivision-set-reconstructible if either (i) G has no stretches but has an edge with neither vertex of degree 2, or (ii) the lengths of the stretches in G are bounded.

Proof : G satisfies the hypothesis of the theorem iff the lengths of the stretches seen in all the e.ss are bounded. So from the e.ss of G, we can determine whether G satisfies the hypothesis of the theorem or not. Now the proof *is* same as that of Result 1.

Corollary – If an infinite graph G has only a finite number of vertices of degree 2, then it is subdivision-set-reconstructible.

A stretch whose ends have degrees m and n in G is called an m, n-stretch. Thus every stretch is either a 2, 2-stretch or an m, n-stretch where $m \neq 2 \neq n$. A 2, 2-stretch is a component that is a cycle. A stretch one of whose ends has degree m, $m \neq 2$ is also called an m-stretch.

Lemma 1 – Let G be an infinite graph not satisfying the hypothesis of Theorem 1. G has an m, n-stretch of length r iff all but at most r of the e.ss of G have an m, n-stretch of legnth r.

Proof : If there exists an m, n-stretch of length r, then this stretch will be as it is when any edge other than the r eges on this m, n-stretch is subdivided. Hence all but at most r or the e.ss have an m, n-stretch of length r.

Conversely, let all but at most r or the e.ss have an m, n-stretch of length r. If possible, let there be no m, n-stretch of length r in G. Now we have all but at most r of the edges of G lying on an m, n-stretch of length $r-1$. Each of the remaining (at most r) edges either lies on no stretch or lies on a stretch of length at most r. Thus the length of a stretch in G is at most r, contradicting the hypothesis of the lemma. This completes the proof.

Theorem 2 – If an infinite graph G has no m, n-stretch of length k but has a shorter m, n-stretch or an edge with degree pair (m, n), then G is subdivision-reconstructible.

Proof : If G satisfies the hypothesis of Theorem 1, then G is subdivision-reconstructible by Theorem 1. Otherwise by Lemma 1 above, for any given pair of integers m and n, we can find out whether G has an m, n-stretch of length r or not for any chosen positive integer r. Also, G has an edge with degree pair (m, n) iff

all but at most one of the e.ss have an edge with degree pair (m, n). Hence if G does not satisfy the condition for subdivision-reconstructibility given in Theorem 1, the situation mentioned in this theorem is recognisable from the collection of e.ss and G has no m, n-stretch of length t for some known $t \geq 2$ such that G has an m, n-stretch of length $t - 1$ if $t > 2$ or an edge with degree pair (m, n) if $t = 2$. Now take an e.s. that has an m, n-stretch of length t. This stretch is unique in this e.s. and suppressing a vertex of this stretch, we get G.

Corollary – If there exists an integer k such that G has no stretch of length k but has a shorter stretch, then G is subdivision-reconstructible.

Lemma 2 – Let G be an infinite graph not satisfying the hypothesis of Theorem 1. Now G has a cycle of length r iff all but at most r of the e.ss of G have a cycle of length r.

Proof : Similar to that of Lemma 1 above.

Theorem 3 – If G has a cycle of length r, but no cycle of length $r + k$ for some integer $k > 0$, then G is subdivision-reconstructible.

Proof : If G satisfies the hypothesis of Theorem 1, then G is subdivision-reconstructible. Otherwise, by Lemma 2 and the hypothesis of the theorem, we can find a number m $(m \geq r)$ such that G has a cycle of length m but none of length $m + 1$. Consider e.ss that have a cycle of length $m + 1$. Choose one among these e.ss in which a cycle of length $m + 1$ contains a stretch of maximal length. This stretch is unique in this e.s. and suppressing a vertex of this stretch, we get G.

Corollary – If G is an infinite bipartite graph having a cycle, then G is subdivision-reconstructible.

Theorem 4 – If each component G_i of an infinite graph G has a vertex of finite degree d_i such that the total number of vertices of degree d_i in G is finite, then G is subdivision-set-reconstructible.

Proof : Since the degree sequence of G can be found from that of any e.s., the situation is recognisable and the number of times each degree occurs is also known. Let K be a subset of non-negative integers such that G has only a known finite number of vertices of degree k for each k in K and each component of G has a vertex whose degree is in K. If $2 \in K$, then by Corollary to Theorem 1, G is subdivision-set-reconstructible. Hence let $2 \notin K$.

Consider k $(\neq 0)$ in K. The number of edges incident with vertices of degree k is finite. Hence if the k-stretches in all the e.ss of G have a finite maximum length m (≥ 2), then G is the graph obtained by suppressing a vertex of a k-skretch of length m in an e.s. having such a stretch. Otherwise, at each vertex of degree k, k infinite stretches meet. Hence each component of G containing a vertex of degree k is precisely k infinite stretches meeting at a vertex.

Thus if a vertex of G whose degree is in K is incident with a stretch in an e.s., then G is uniquely determined from a suitable e.s. Otherwise, the components (with multiplicity) of G corresponding to each degree in K is known. These are precisely

the components of G as in this case, none of the components of G has two representatives in K and each component of G has a represntative in K. Hence G is subdivision-set-reconstructible.

Theorem 5 – If G has a nontrivial finite component and the orders of such components are bounded, then G is subdivision-set-reconstructible.

PROOF : Let r be the maximum order of a finite component in any e.s.. By hypothesis, r exists and is finite. Let k be the maximum length of a stretch occurring in a component of order r in any e.s.. Choose an e.s. that has a component of order r having a stretch of length k. This component of order r is unique in this e.s. and the stretch of length k is unique in this component. Suppressing a vertex of this stretch, we get G.

Theorem 6 – Every tree with at least two but only finitely many vertices of infinite degree is subdivision-set-reconstructible.

PROOF : There are finite number (≥ 2) of vertices of infinite degree and they can be located in each e.s.. Between any two such vertices, there exists a unique path. Take an e.s. that has a stretch of maximum length forming part of the path between any two vertices of infinite degree. Since there are only a finite number of such vertices, this maximum length is finite and the corresponding stretch is unique in this e.s. Suppressing a vertex of this stretch we get G.

We believe that every infinite graph is subdivision-reconstructible.

Corresponding results hold for directed graphs also.

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REFERENCE

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, 1976.