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FIXED POINT THEOREM FOR DENSIFYING MAPPINGS IN COMPLETE METRIC SPACE

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In this paper the results of Fisher[3], Khan[6], Furi and Vignoli[2] and Iseki[4] have been generalized to a more general case for a pair of densifying mappings in bounded complete metric space. For related results we refer to the recent paper of Pande[8].

Kuratowski[5] define the measure of non-compactness of α as follows:

Let A be a bounded subset of a metric space X. By the real number $\alpha(A)$, we denote the infimum of all positive number ϵ such that A admits a finite covering consisting of subsets with diameters less than ϵ. Then we have the following results:

(i) $0 \leq \alpha(A) \leq \delta(A)$, where $\delta(A)$ is the diameter of A.

(ii) $\alpha(A) = 0$, iff A is precompact,

(iii) $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$.

For some fundamental properties about measure of non-compactness α of bounded sets in a metric space, see Nussbaum[7].

The following notion of densifying mapping was introduced by Furi and Vignoli[2].

Definition – Let (X, d) be a metric space. T be a continuous mapping of X into itself. The mapping T is called densifying, if for every bounded subsets A of X with $\alpha(A) \geq 0$ we have $\alpha(T(A)) < \alpha(A)$.

Here we shall prove the following theorem:

Theorem – Let S and T are two densifying mappings of a bounded complete metric space (X, d) satisfying,

$$\text{(I)} \quad d(Sx, STy) < \alpha_1 d(x, Ty) + \alpha_2 \{d(x, Sx) + d(Ty, STy)\} + \alpha_3 \{d(x, STy) + d(Ty, Sx)\}$$

for all x, y in X with $x \neq Ty$

$$\text{(II)} \quad ST = TS$$

where α_1, α_2 and α_3 are non-negative real constants and $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$, then S and T have a common fixed point in X, which is unique whenever $\alpha_1 + 2\alpha_3 = 1$.

PROOF : Let x_0 be a point of X, the sequence $\{x_n\}$ defined by

$$Sx_{2n} = x_{2n+1}, \; Tx_{2n+1} = x_{2n+2}, \text{ for } n = 0, 1, 2. \ldots\ldots$$

Put $A = \{x_{2n+1} : n = 0, 1, 2, \ldots\}$ then,

$ST(A) = \{x_{2n+1} : n = 1, 2, \ldots\}$ and so $ST(A) \subset A$. By the continuity of S and T, $ST(\bar{A}) \subset \overline{ST(A)} \subset \bar{A}$. Hence $\bar{A}$ is invariant under ST and is bounded. Suppose $\alpha(A) > 0$.

Since $A = ST(A) \cup \{x_1\}$, we have,

$$\alpha(A) = \max(\alpha(ST(A)), \alpha(x_1))$$

$$= \alpha(ST(A)) < \alpha(A), \text{ a contradiction.}$$

Since the mappings S and T are densifying. Thus $\alpha(A) = 0$, which implies that A is precompact. Since X is complete metric space, $\bar{A}$ is compact. Define a real valued function f on X by $f(x) = d(Tx, STx)$. By the hypothesis, d is continuous on the compact subset $\bar{A}$. Hence d has a minimum point u in $\bar{A}$. We now affirm that u is a fixed point for S. If not let us suppose that $Su \neq u$. Then using (I) and (II), we have,

$$\begin{aligned} f(Su) &= d(TSu, STSu) \\ &= d(STu, STSu) \\ &< \alpha_1 d(Tu, TSu) + \alpha_2 \{d(Tu, STu) + d(TSu, STSu)\} \\ &\quad + \alpha_3 \{d(Tu, STSu) + d(TSu, STu) \\ &= \{(\alpha_1 + \alpha_2 + \alpha_3) d(Tu, STu) + (\alpha_2 + \alpha_3) d(TSu, STSu)\} \end{aligned}$$

i.e. $$d(STu, STSu) < \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3} d(Tu, STu).$$

$d(STu, STSu) < d(Tu, STu)$, a contradiction. Hence $u \in X$ is a fixed point for S that is $Su = u$. We have,

$$STu = TSu = = Tu.$$

Now we shall prove that $Tu = u$. If not, let us suppose that $Tu \neq u$ then we have by (I),

$$d(u, Tu) = d(Su, STu)$$

$$< \alpha_1 d(u, Tu) + \alpha_2 \{d(u,u) + d(Tu, Tu)\} + \alpha_3 \{d(u, Tu) + d(Tu, u)\}$$

$d(u, Tu) < d(u, Tu)$, again leading to a contradiction. This contradiction proves that $Tu = u$.

To prove the uniqueness, if possible let v be the another fixed point of $T(u \neq v)$, then

$$d(u, v) = d(Su, STv)$$

$$< \alpha_1 d(u, v) + \alpha_2 \{d(u, u) + d(v, v)\} + \alpha_3 \{d(u, v) + d(v, u)\}$$

$$d(u, v) < (\alpha_1 + 2\alpha_3) d(u, v)$$

$d(u, v) < d(u, v)$, a contradiction. Therefore u is a unique common fixed point. This completes the proof of the theorem.

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