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ON PERIODIC ORBITS IN THE CIRCULAR RESTRICTED PROBLEM OF THREE BODIES IN A THREE-DIMENSIONAL COORDINATE SYSTEM

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In this paper a criterion for the existence of periodic orbits of a circular restricted problem of three bodies in three-dimensional coordinate system has been established applying variational principle and verified with the help of numerical example.

1. Introduction

The problem discussed in this paper is a generalization of Whittaker's work[3] in which the existence of periodic orbits of the plane circular restricted problem of three bodies has been studied. In this paper the existence of periodic orbits of the circular restricted problem of three bodies in a three-dimensional coordinate system has been studied. With the introduction of the third coordinate the problem becomes complicated.

In the second section of the paper the equations of motion in the canonical form has been given and in the third section the variation of the time integral of the Lagrangian function has been discussed for the existence of the periodic orbits. Due to the presence of the third coordinate the criterion for the existence of periodic orbits has been obtained in a slightly different manner than that of Whittakar[3]. In the fourth and the last section of this paper we have constructed a numerical example of the problem. For this we have chosen the orbit of the periodic Comet Harrington. Runge-Kutta's method of numerical integration has been followed here. All the criteria for the existence of periodic orbits obtained in the third section are shown to be satisfied. Thus the existence of periodic orbit is shown for the numerical example discussed in the fourth section.

2. Equation of Motion

Let m_1 and m_2 be the masses of the two finite bodies called primaries which are moving in circular orbits around their centre of mass. Let us take a coordinate system with origin at the centre of the finite masses and common orbital plane of the finite bodies as the xy-plane and the line joining the two finite masses as the X-axis which

rotates in the xy-plane in the direction in which the finite bodies move with the constant angular velocity n about z-axis. If (x, y, z) be the coordinates of the infinitesimal body referred to the above rotating system of coordinate axes, the equations of motion may be given in the canonical form

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p_1}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial p_2}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial p_3} \\ \frac{dp_1}{dt} &= -\frac{\partial H}{\partial x}, \quad \frac{dp_2}{dt} = -\frac{\partial H}{\partial y}, \quad \frac{dp_3}{dt} = -\frac{\partial H}{\partial z} \end{aligned}\right\} \qquad \ldots(2.1)$$

where

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + p_3^2 + 2n\,(yp_1 - xp_2)\right] - k^2\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)$$

with

$$\left.\begin{aligned} p_1 &= \dot{x} - ny \\ p_2 &= \dot{y} + nx \\ p_3 &= \dot{z} \end{aligned}\right] \qquad \ldots(2.2)$$

and k the Gaussian constant, and r_1 and r_2 the distances of the finite masses m_1 and m_2 respectively from the infinitesimal mass.

Here the integral of energy is

$$H = C. \qquad \ldots(2.3)$$

3. A criterion for the Existence of Periodic Orbits

Let us consider the integral

$$J = \int_{t_0}^{t_1} L\,dt$$

$$= \int_{t_0}^{t_1} (-H + p_1\dot{x} + p_2\dot{y} + p_3\dot{z})\,dt. \qquad \ldots(3.1)$$

The variation of this integral between fixed limits vanishes along an orbit since it is a minimum or at least stationary along an orbit as compared with its value along any neighbouring path. Using (2.2) and (2.3) the integral (3.1) becomes

$$J = \int_0^1 \left\{ v\,ds + n\,(x\,dy - y\,dx)\right\} - \int_{t_0}^{t_1} C\,dt$$

where 0 and 1 stand for the values of S, x, y etc. corresponding to $t = t_0$ and $t = t_1$,

respectively. Now if $\delta x = \delta y = \delta z = 0$ at the limits,

$$\delta \int_0^1 v ds = \int_0^1 \left\{ \delta v ds - d\left(v \frac{dx}{ds}\right) \delta x - d\left(v \frac{dy}{ds}\right) \delta y - d\left(v \frac{dz}{ds}\right) \delta z \right\} + \left[v \frac{du}{ds} \delta u + v \frac{dy}{ds} \delta y + v \frac{dz}{ds} \delta z \right]_0^1$$

and

$$\delta \int_0^1 n\,(xdy - ydx) = 2n \int_0^1 (\delta x dy - \delta y\, dx) \text{ and } \delta \int_{t_0}^{t_1} C\, dt = 0.$$

Therefore

$$\delta J = \int_0^1 \left(\delta v ds - \delta x\, d\left(v \frac{dx}{ds}\right) - \delta y\, d\left(v \frac{dy}{ds}\right) - \delta z\, d\left(v \frac{dz}{ds}\right) \right.$$

$$\left. 2n\,(\delta x\, dy - \delta y\, dx). \right. \qquad \ldots(3.2)$$

Let δp be the length of the principal normal at (x, y, z) between the curve and an outer neighbouring curve. The direction cosines of the tangent and the principal normal to the curve at (x, y, z) are, respectively,

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \text{ and } -\rho \frac{d_2x}{ds^2}, -\rho \frac{d^2y}{ds^2}, -\rho \frac{d^2z}{ds^2};$$

ρ being the radius of curvature. (The direction of the principal normal to be taken positive when along the concave side).

Hence

$$\delta x = -\delta p\, \rho \frac{d^2x}{ds^2}$$

$$\delta y = -\delta p\, \rho \frac{d^2y}{ds^2} \qquad \ldots(3.3)$$

$$\delta z = -\delta p\, \rho \frac{d^2z}{ds^2}.$$

Therefore

$$\delta x\, d\left(v \frac{dx}{ds}\right) + \delta y\, d\left(v \frac{dy}{ds}\right) + \delta z\, d\left(v \frac{dz}{ds}\right) = -\frac{v}{\rho} \delta p\, ds \ldots(3.4)$$

and

$$2n\,(\delta x\, dy - \delta y\, dx) = -2n\rho\, \delta p\, ds \left(\frac{d^2x}{ds^2} \frac{dy}{ds} - \frac{d^2y}{ds^2} \frac{dx}{ds}\right). \qquad \ldots(3.5)$$

Using (3.4) and (3.5), the integral (3.2) becomes

$$\delta J = \int_0^1 K\, \delta p\, ds$$

where

$$K = \frac{\partial v}{\partial p} + \frac{v}{\rho} + 2n\rho\left(\frac{d^2y}{ds^2}\frac{dx}{ds} - \frac{d^2x}{ds^2}\frac{dy}{ds}\right).$$

The quantity δJ can be easily calculated at any instant on a given curve. Therefore, as in (Whittaker[4] § 168) we conclude that if one closed curve be enclosed by another closed curve and if the quantity δJ is positive at all points on one of the two curves and negative at all points on the other curve, then in the ring-shaped space between the two curves there exists a periodic orbit of the dynamical system.

4. Numerical Application

In order to examine the existence of periodic orbits in a circular restricted problem of three bodies in a three-dimensional coordinate system by means of a numerical example, we have considered the orbit of a periodic Comet Harrington, using the orbital elements given in *I.A.U. Circular No.* 1713 as follows:

τ = 1960 June 28.8327; e = 0.559273; a = 3.590373;
w = 232°.8391; Ω = 119°.1327; I = 8°.6838; Equinox 1950.0. ...(4.1)

Using the notations and the computations given by Brouwer and Clemence[1] (§ 24), the three-dimensional rectangular equitorial coordinates and the velocity components of the periodic Comet Harrington on 1960 June 5.0 are given by

x = 1.4673 y = - 0.5112 x = - 0.3847 ...(4.1)

and $\dot{x}$ = 0.0049504 $\dot{y}$ = 0.0155140 $\dot{z}$ = 0.0047036. ...(4.3)

Let the masses of the earth and the sun as given by Escobal[2], (Table 1, 3) be

$m_1 = 2.9991 \times 10^{-6}$ and $m_n = 1$, respectively.

Let the units of length and time be the distance between the sun and the earth and the mean solar day, respecitvely. Therefore, the coordinates of the sun and the earth with their centre of mass as the origin and the line joining them as the axis of x and their common orbital plane as the xy-plane are

$(-2.9991 \times 10^{-6}, 0, 0)$ and $(+ 0.999997, 0, 0)$,

respectively. Hence the coordinates of the periodic Comet Harrington referred to the above system of coordinate axes on 1960 June 5.0 are given by

$$x = 1.467297, \quad y = - 0.511200, \quad z = - 0.384700, \quad \ldots(4.4)$$

With the sun and the earth moving in circular orbits about their centre of mass as the finite masses and the Comet as the infinitesimal mass, the motion of the Comet will be treated as the circular restricted problem of three bodies in a three-dimensional coordinate system and the equations of motion of the problem will be given by

$$\ddot{x} - 2n\dot{y} - n^2x = \frac{\partial w}{\partial x}$$

$$\ddot{y} + 2n\dot{x} - n^2 y = \frac{\partial w}{\partial y}$$

$$\ddot{z} = \frac{\partial w}{\partial z} \qquad \ldots(4.5)$$

with $$w = k^2 \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right)$$

where K is the Gaussian Constant, and r_1 and r_2 are the distances of the Comet from the earth and the sun, respectively and n the mean motion or the constant angular velocity of the earth in its circular orbit with origin at the centre is equal to 0.017202 [Escobal[2], § 1.3.1]. Using (2.2) from (3.1) we have the differential equation

$$\frac{dJ}{dt} = F \qquad \ldots(4.6)$$

where $$F = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + n(x\dot{y} - y\dot{x}) + \tfrac{1}{2} n^2 (x^2 + y^2) + k^2 \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right).$$

We have solved differential equations (4.5) and (4.6) by Runge-Kutta's method of numerical integration for the actual orbit of the Comet Harrington when on $t = t_0 = 0$ (1960 June 5.0) the initial values of x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$ are given by (4.4) and (4.3) respectively, and when $t_1 = t_0$ (1960 June 5.0) we have from (3.1) $J = 0$. For the actual orbit the values of x, y, z and J thus obtained on different dates are tabulated in Table I. Next we have solved the differential equations (4.5) and (4.6) by Runge-Kutte's method for two adjacents paths, namely adjacent path No. 1 and adjacent path No. 2 of the Comet by giving some variations with the initial value of the x-coordinate in (4.4) only. We take $x = 1.465297$ for the adjacent path No. 1 and $x = 1.469297$ for the adjacent path No. 2, other initial values remaining same as those for the actual orbit. The values of x, y, z and J obtained for the two adjacent path No. 1 and 2 are separately tabulated in Table II and III respectively.

Comparing the values of J obtained in the three alone cases the variations in J denoted by δJ for the adjacent paths tabulated in Table IV. We see that δJ is positive on adjacent path No. 1 and δJ is negative on adjacent path No. 2. Thus all the conditions for the existence of periodic orbits given in the concluding lines of section three of this paper are completely satisfied. Therefore, there exists a periodic orbit of the comet in the ring shaped space between the two adjacent paths. In the numerical example of this section actual orbit of the Comet is the periodic orbit.

TABLE I

t in days		x	y	z	J
June 5.0, 1960	0	1.467297	- 0.511200	- 0.384700	0.0
	10	1.558204	- 0.374558	- 0.336342	0.0084906758
	20	1.723549	- 0.291062	- 0.285737	0.0174871175
	30	1.947337	- 0.282233	- 0.233599	0.0261688700
	40	2.209108	- 0.365052	- 0.180555	0.0339253942
	50	2.484486	- 0.551111	- 0.127024	0.0406324242

TABLE II

t in days		x	y	z	J
June 5.0, 1960	0	1.465297	- 0.511200	- 0.384700	0.0
	10	1.556164	- 0.374547	- 0.336337	0.0084959683
	20	1.721394	- 0.291005	- 0.285720	0.0175083191
	30	1.945007	- 0.282071	- 0.233564	0.0262075749
	40	2.206567	- 0.364712	- 0.180498	0.0339867607
	50	2.481720	- 0.550502	- 0.126945	0.0407103007

TABLE III

t in days		x	y	z	J
June 5.0, 1960	0	1.469297	- 0.511200	- 0.384700	0.0
	10	1.560244	- 0.374568	- 0.336347	0.0084856167
	20	1.725704	- 0.291119	- 0.285754	0.0174657100
	30	1.949666	- 0.282394	- 0.233634	0.0261192158
	40	2.211650	- 0.365393	- 0.180611	0.0338554441
	50	2.487251	- 0.551720	- 0.127103	0.0405472666

TABLE IV

t in days		J	
		Adjacent path No. 1	Adjacent path NO.2
June 5.0, 1960	0	0	0
	10	+0.0000052925	- 0.0000050591
	20	+0.0000212016	- 0.0000214075
	30	+0.0000387049	- 0.0000496542
	40	+0.0000613685	- 0.0000699501
	50	+0.0000778765	- 0.0000851576

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