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CERTAIN INEQUALITIES FOR APPELL'S F_3 AND F_4

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Inequalities for F_4 for positive real arguments are obtained. Although the inequalities obtained are not very sharp but they fill certain gaps in the existing literature. In the sequel an inequality for F_3 is also obtained.

1. INTRODUCTION

Two-sided inequalities for Appell's functions F_1, F_2 and F_3 had been obtained by Luke[6] for negative real argument through their Eulerian integral representations under suitable restrictions on parameters and variables. Joshi and Arya[4] had obtained the two-sided inequalities for F_1 and F_2 by applying theorems of Flett[3] and Carlson[2], for positive real argument. But so far no inequality for F_4 is available in the literature. This motivated us to obtain two-sided inequalities for F_4 for positive real argument and in the sequel also obtain an inequality for F_3. Though these results are not very sharp from computational view point, but they fill certain gaps in the literature on inequalities of these functions.

We recall here the theorem of Flett[3], which will be used in the investigation that follows:

Theorem 1 – Let $a > c > b > 0$, $0 < x < 1$, then

$$\frac{\Gamma(a + b - c)\,\Gamma c}{\Gamma a\,\Gamma b}\,(1 - x)^{c-a-b} < {}_2F_1\,(a, b; c; x) < (1 - x)^{c-a-b}.$$

Theorem 2 – Let $c > a > c - b > 0$, $b > 0$, $0 < x < 1$,

then

$$(1 - x)^{c-a-b} < {}_2F_1\,(a, b; c; x) < \frac{\Gamma(a + b - c)\,\Gamma c}{\Gamma a\,\Gamma b} < (1 - x)^{c-a-b}.$$

2. INEQUALITIES FOR F_4

Consider the expansion formula for F_4 due to Burchnall and Chaundy[1] (see also Srivastava and Karlsson[7], p. 332).

$$F_4\ (a,\ b;\ c_1,\ c_2;\ x\ (1-y),\ y(1-x))$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\ (b)_r\ (a+b-c_1-c_2+1)_r\ x^r\ y^r}{r!\ (c_1)_r\ (c_2)_r}$$

$$\times\ {}_2F_1\ (a+r,\ b+r;\ c_1+r;\ x)\ {}_2F_1\ (a+r,\ b+r;\ c_2+r;\ y) \quad \ldots(2.1)$$

where F_4 is defined by

$$F_4\ (a,\ b;\ c_1,\ c_2;\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\ (b)_{m+n}\ x^m\ y^n}{(c_1)_m\ (c_2)_n\ m!\ n!}$$

$|\sqrt{x}| + |\sqrt{y}| < 1$, neither c_1 nor c_2 is a negative integer or zero. $\ldots(2.2)$

$$a > c_1 > b > 0,\ a > c_2 > b > 0,\ 0 < x < 1,\ 0 < y < 1,$$

$$\sqrt{(x\ (1-y))} + \sqrt{(y\ (1-x))} < 1 \quad \ldots(2.3)$$

application of Theorem 2 to the ${}_2F_1$'s leads to

$$(1-x)^{c_1-a-b}\ (1-y)^{c_2-a-b}\ \frac{\Gamma(a+b-c_1)\ \Gamma(a+b-c_2)\ \Gamma c_1\ \Gamma c_2}{(\Gamma a)^2\ (\Gamma b)^2}$$

$$\times\ {}_3F_2\left[\begin{matrix} a+b-c_1,\ a+b-c_2,\ a+b-c_1-c_2+1; \\ a, \qquad\qquad b; \end{matrix}\ \frac{x\ y}{(1-x)(1-y)}\right]$$

$$< F_4\ (a,\ b;\ c_1,\ c_2;\ x\ (1-y),\ y(1-x))$$

$$< (1-x)^{c_1-a-b}\ (1-y)^{c_2-a-b}\ {}_3F_2\left[\begin{matrix} a,\ b;\ a+b-c_1-c_2+1, \\ c_1, \qquad\quad c_2 \end{matrix}\ \frac{x\ y}{(1-x)(1-y)}\right]$$

$\ldots(2.4)$

Further estimation of ${}_3F_2$'s by Luke's result[5] (4.2)

$$(1-\theta z)^{-a} < {}_{p+1}F_p\left(\begin{matrix} a;\ (\alpha_p)\ ; \\ (\beta_p)\ ; \end{matrix}\ z\right) < 1 - \theta + \theta\ (1-z)^{-a}$$

$$0 < z < 1,\ a > 0,\ \beta_p \geq \alpha_p > 0,\ \theta = \pi\ \frac{\alpha_p}{\beta_p} \quad \ldots(2.5)$$

in conjunction with the symmetry relation

$$F_4(a,\ b;\ c_1,\ c_2;\ x(1-y),\ y(1-x)) = F_4(a,\ b;\ c_2,\ c_1;\ y(1-x),\ x(1-y)) \quad \ldots(2.6)$$

gives rise to the following

Theorem 3 – Let $a > c_1 > b > 0,\ a > c_2 > b > 0,$

$$(a+1) \leq c_1 + c_2 \leq a + b + 1 \leq \min\{c_1 + 2c_2,\ c_2 + 2c_1\},$$

$$0 < x < 1,\ 0 < y < 1,\ \sqrt{(x\ (1-y))} + \sqrt{(y\ (1-x))} < 1$$

then

$$(1-x)^{c_1-a-b}\ (1-y)^{c_2-a-b}\ \frac{\Gamma(a+b-c_1)\ \Gamma(a+b-c_2)\ \Gamma c_1\ \Gamma c_2}{(\Gamma a)^2\ (\Gamma b)^2}$$

$$\text{Max}\left\{\begin{array}{l}\left[1-\dfrac{(a+b-c_1)\ (a+b-c_2-c_1+1)\ xy}{ab\ (1-x)\ (1-y)}\right]^{c_2-a-b}\\[2ex]\left[1-\dfrac{xy(a+b-c_2)\ (a+b-c_1-c_2+1)}{ab\ (1-x)\ (1-y)}\right]^{c_1-a-b}\end{array}\right.$$

$$< F_4\ (a,\ b;\ c_1,\ c_2;\ x\ (1-y),\ y(1-x))$$

$$< (1-x)^{c_1-a-b}\ (1-y)^{c_2-a-b}\left[1-\frac{b(a+b-c_1-c_2+1)}{c_1\ c_2}\left\{1-\left[1-\frac{xy}{(1-x)\ (1-y)}\right]^{-a}\right\}\right]$$

Another theroem in the case when both c_1 and c_2 are greater then a and b can be obtained by using Theorem 1 instead of Theorem 2. Thus in a similar fashion we have:

Theorem 4 — Let $c_1 > a > c_1 - b > 0$, $c_2 > a > c_2 - b > 0$,

$$\max\{a+1,\ b+1\} \le c_1 + c_2 \le a + b + 1 \le \min\{c_1+2c_2,\ c_2+2c_1\},$$

$$0 < x < 1,\ 0 < y < 1,\ \sqrt{(x\ (1-y))} + \sqrt{(y\ (1-x)} < 1$$

then

$$(1-x)^{c_1-a-b}\ (1-y)^{c_2-a-b}\ \text{Max}\left\{\begin{array}{l}\left[1-\dfrac{ab\ xy}{c_1\ c_2\ (1-x)\ (1-y)}\right]^{c_1+c_2-a-b-1}\\[2ex]\left[1-\dfrac{(a+b-c_1-c_2+1)\ bxy}{c_1\ c_2\ (1-x)\ (1-y)}\right]^{-a}\\[2ex]\left[1-\dfrac{(a+b-c_1-c_2+1)\ axy}{c_1\ c_2\ (1-x)\ (1-y)}\right]^{-b}\end{array}\right.$$

$$< F_4\ (a,\ b;\ c_1,\ c_2;\ x\ (1-y),\ y(1-x))$$

$$< (1-x)^{c_1-a-b}\ (1-y)^{c_2-a-b}\ \frac{\Gamma(a+b-c_1)\ \Gamma(a+b-c_2)\ \Gamma c_1\ \Gamma c_2}{(\Gamma a)^2\ (\Gamma b)^2}$$

$$\times \min \left[\begin{array}{l} 1 - \dfrac{(a+b-c_1)\,(a+b-c_2)}{ab} + \dfrac{(a+b-c_1)\,(a+b-c_2)}{ab} \left[1 - \dfrac{xy}{(1-x)\,(1-y)}\right]^{c_1+c_2-a-b-1} \\ 1 - \dfrac{(a+2b-2c_1-c_2+1)\,(a+2b-c_1-2c_2+1)}{b\,(a+b-c_1-c_2+1)} \left[1 - \left\{1 - \dfrac{xy}{(1-x)\,(1-y)}\right\}^{-a}\right] \\ 1 - \dfrac{(2a+b-2c_1-c_2+1)(2a+b-c_1-2c_2+1)}{a\,(a+b-c_1-c_2+1)} \left[1 - \left\{1 - \dfrac{xy}{(1-x)\,(1-y)}\right\}^{-b}\right]. \end{array} \right.$$

In particular, for the set of parameters and variables

$$a = 3,\ b = 2,\ c_1 = 2.5,\ c_2 = 2.4,\ x = 0.2,\ y = 0.3$$

from Theorem 3 we have

$$3.9798552 < F_4\,(a, b; c_1, c_2; x\,(1-y)\,y(1-x)) < 5.0715551,$$

while for

$$a = 1.6,\ b = 1.4,\ c_1 = 2,\ c_2 = 1.7,\ x = 0.2,\ y = 0.4$$

Theorem 4 gives

$$2.522660 < F_4\,(a, b; c_1; c_2; x\,(1-y),\ y(1-x)) < 3.2352291$$

3. An Inequality for F_3

Under the restrictions

$$c > a_1 > c - b_1 > 0,\ c > a_2 > c - b_2 > 0,\ 0 < x < 1,\ 0 < y < 1 \qquad \ldots(3.1)$$

application of Theorem 1 to Burchnall and Chaundy's[1] expansion formula

$$F_3 \equiv F_3\,(a_1, a_2, b_1, b_2; c; x, y)$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r\,(a_2)_r\,(b_1)_r\,(b_2)_r\,x^r\,y^r\,(-1)^r}{r!\,(c+r-1)_r\,(c)_{2r}}$$

$$\times\ {}_2F_1\left(\begin{matrix} a_1 + r,\ b_1 + r; \\ c + 2r; \end{matrix}\ x\right) {}_2F_1\left(\begin{matrix} a_2 + r,\ b_2 + r\,; \\ c + 2r\,; \end{matrix}\ y\right) \qquad \ldots(3.2)$$

where

$$F_3 = \sum_{m,n=0}^{\infty} \frac{(a_1)_m\,(a_2)_n\,(b_1)_m\,(b_2)_n\,x^m y^n}{(c)_{m+n}\,m!\,n!}$$

$$|x| < 1,\ |y| < 1,\ c \text{ is not a negative integer or zero} \qquad \ldots(3.3)$$

gives

$$(1-x)^{c-a_1-b_1}\,(1-y)^{c-a_2-b_2}\,{}_5F_4\left[\begin{matrix} a_1,\ a_2,\ b_1,\ b_2,\ c-1\ ; \\ \dfrac{c+1}{2},\ \dfrac{c}{2},\ \dfrac{c}{2},\ \dfrac{c-1}{2}\ ; \end{matrix}\ -\frac{xy}{16}\right] < F_3$$

$$< \frac{\Gamma(a_1+b_1-c)\,\Gamma(a_2+b_2-c)\,(\Gamma c)^2}{\Gamma a_1\,\Gamma b_1\,\Gamma a_2\,\Gamma b_2}$$

$$(1-x)^{c-a_1-b_1}\,(1-y)^{c-a_2-b_2}\,{}_2F_1\left[\begin{matrix} c-1,\ (c+1)/2, \\ (c-1)/2, \end{matrix}\ -xy\right]. \qquad \ldots(3.4)$$

Expressing ${}_5F_4$ and ${}_2F_1$ as

$${}_5F_4\left[\begin{matrix} a_1,\ a_2,\ b_1,\ b_2,\ c-1\ ; \\ \dfrac{c-1}{2},\ \dfrac{c}{2},\ \dfrac{c}{2},\ \dfrac{c+1}{2}\ ; \end{matrix}\ -\frac{xy}{16}\right]$$

$$= {}_6F_5\left[\begin{matrix} 1,\ c-1,\ a_1,\ a_2,\ b_1,\ b_2\ ; \\ \dfrac{c-1}{2},\ \dfrac{c}{2},\ \dfrac{c}{2},\ \dfrac{c+1}{2},\ 1\ ; \end{matrix}\ -\frac{xy}{16}\right] \qquad \ldots(3.5)$$

$${}_2F_1\left[\begin{matrix} \dfrac{c+1}{2},\ c-1\ ; \\ \dfrac{c-1}{2}\ ; \end{matrix}\ -xy\right] = {}_3F_2\left[\begin{matrix} 1,\ \dfrac{c+1}{2},\ c-1\ ; \\ \dfrac{c-1}{2},\ 1\ ; \end{matrix}\ -xy\right] \qquad \ldots(3.6)$$

and applying Luke's result[5] (4.12)

$$1 - \frac{a(c+1)}{c(a+1)} + \frac{a(c+1)}{c(a+1)}\left[1 + \frac{(a+1)\,\theta z}{(c+1)}\right]^{-1}$$

$$\leqq {}_{p+2}F_{p+1}\left[\begin{matrix} 1,\ a,\ \alpha p\ ; \\ c,\ \beta p\ ; \end{matrix}\ -z\right] < 1 - \frac{\theta}{\phi} + \frac{\theta}{\phi}\left[1 + \frac{a\phi z}{c}\right]^{-1}$$

$$z > 0,\ 0 < c \leq a,\ \beta_p \geq \alpha_p > 0,\ \theta = \pi\,\frac{\alpha_p}{\beta_p},\ \phi = \pi\,\frac{(\alpha_p+1)}{(\beta_p+1)} \qquad \ldots(3.7)$$

we have :

Theorem 5 – Let $2 \geq c \geq \text{Max}\,\{1, 2b_1, 2b_2, 2a_1 - 1\}$, $a_2 \leq 1$, $c > a_1 > c - b_1 > 0$, $c > a_2 > c - b_2 > 0$, $b_1 > 0$, $b_2 > 0$, $0 < x < 1$, $0 < y < 1$

then

$$(1-x)^{c-a_1-b_1}(1-y)^{c-a_2-b_2}\left[1-\frac{c+1}{c}+\frac{c+1}{c}\left\{1+\frac{a_1 a_2 b_1 b_2 xy}{c(c+1)^2}\right\}^{-1}\right]$$

$$< F_3 < \frac{\Gamma(a_1+b_1-c)\,\Gamma(a_2+b_2-c)\,(\Gamma c)^2}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(b_1)\,\Gamma(b_2)}(1-x)^{c-a_1-b_1}(1-y)^{c-a_2-b_2}$$

$$\times\left[1-\frac{2(c-1)}{c}+\frac{2(c-1)}{c}\left\{1+\frac{c(c+1)\,xy}{2(c-1)}\right\}^{-1}\right]. \qquad \ldots(3.8)$$

It is interesting to note that Theorem 5 is symmetric in the sense of F_3, i.e. interchange of a_1, b_1 and x with a_2, b_2 and y, respectively does not affect the inequality (3.8). So, if we want to obtain inequality when $a_1 < 1$, $a_2 > 1$, we have to replace a_1, b_1, x by a_2, b_2, y respectively and vice-versa.

For the set of values

$$a_1 = 1.1,\ a_2 = 0.9,\ b_1 = 0.6,\ b_2 = 0.6,\ c = 1.3,\ x = 0.7,\ y = 0.5$$

we have from (3.8)

$$1.8007284 < F_3 < 4.7799066. \qquad \ldots(3.9)$$

References

1. J. L. Burchnall and T. W. Chaundy, *Quart. J. Math.* **11** (1940), 249-70.
2. B. C. Carlson, *Proc. Am. Math. Soc.* **17** (1966), 32-39.
3. T. M. Flett, *Proc. Edinburgh Math. Soc.* (2) **18** (1972), 31-34.
4. C. M. Joshi and J. P. Arya, *Jñānābha* **9/10** (1980), 179-86.
5. Y. L. Luke, *J. Approx. Theory* **5** (1972), 41-65.
6. Y. L. Luke, *J. Approx. Theory* **11** (1974), 73-84.
7. H. M. Srivastava and Per W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood, Chichester, U.K.; John Wiley, New York, 1985.