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DECOMPOSITION OF *-HOMOMORPHISMS OF UNBOUNDED OPERATOR ALGEBRAS

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We shall introduce a class of unbounded operator algebras called regular O^*-algebras which is a wider class than EW^*-algebras and closed O^*-algebras satisfying condition (I), and show that every *-homomorphism Φ of a closed O^*-algebra $\mathfrak{M}$ onto a regular O^*-algebra $\mathfrak{N}$ with a regular basis $\{\eta_\lambda\}_{\lambda\in\Lambda}$ such that $\omega_{\eta_\lambda} \circ \Phi$ is a σ-vector form on $\mathfrak{M}$ for each $\lambda \in \Lambda$ is composed of an ampliation, an induction and a spatial isomorphism. This is an extension of the results of Inoue[5] and Bhatt[2].

1. INTRODUCTION

The purpose of this paper is to extend the well-known result[3] on von Neumann algebras : "Every normal *-homomorphism of a von Neumann algebra $\mathfrak{M}$ onto a von Neumann algebra $\mathfrak{N}$ is composed of an ampliation, an induction and a spatial isomorphism" to unbounded operator algebra (O^*-algebras). The difficulty of this problem due to what there are some pathologies between invariant subspaces $\mathcal{E}$ for O^*-algebras and the projections onto $\bar{\mathcal{E}}$. Inoue[5] and Bhatt[2] extended the above composition theorem on von Neumann algebras to EW^*-algebras and closed O^*-algebras satisfying condition (I) which don't spring up the pathologies, respectively.

In this paper we first show that every *-homomorphism Φ of a closed O^*-algebra $\mathfrak{M}$ onto a self-adjoint O^*-algebra $\mathfrak{N}$ with a strongly cyclic vector η_0 such $\omega\eta_0 \circ \Phi$ is a σ-vector form on $\mathfrak{M}$ is composed of an ampliation, an induction and a spatial isomorphism. Furthermore, we shall extend this result to regular O^*-algebras which are generalization of self-adjoint O^*-algebras with strongly cyclic vector. This is an extension of the results of Inoue[5] and Bhatt[2].

2. PRELIMINARIES

For the sake of completeness we recall in this section some of the definitions and the basic properties of O^*-algebras, and refer to the papers[4,7,8] for further details.

Let $\mathfrak{D}$ be a pre-Hilbert space and $\mathcal{H}(\mathfrak{D})$ the completion of $\mathfrak{D}$. Let $\mathcal{L}^\dagger(\mathfrak{D})$ be the set of all linear operators X from $\mathfrak{D}$ into $\mathfrak{D}$ satisfying $\mathfrak{D}(X^*) \supset \mathfrak{D}$ and $X^*\mathfrak{D} \subset \mathfrak{D}$. Then $\mathcal{L}^\dagger(\mathfrak{D})$ is *a* *-algebra with the usual operations and the involution $X^\dagger = X^* \lceil \mathfrak{D}$. A *-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathfrak{D})$ is said to be an O^*-algebra on $\mathfrak{D}$. Let $\mathfrak{M}$ be an O^*-algebra on $\mathfrak{D}$. A locally convex topology on $\mathfrak{D}$ defined by a family $\{\| \ \|_X; X \in \mathfrak{M}\}$ of seminorms :

$$\|\xi\|_X = \|\xi\| + \|X\xi\|, \ \xi \in \mathfrak{D}$$

is said to be the induced topology and is denoted by $t_{\mathfrak{M}}$. If the locally convex space $\mathfrak{D}[t_{\mathfrak{M}}]$ is complete, then $\mathfrak{M}$ is called closed. We denote by $\tilde{\mathfrak{D}}(\mathfrak{M})$ the completion of $\mathfrak{D}[t_{\mathfrak{M}}]$ and put

$$\tilde{X}\xi = \bar{X}\xi, \ X \in \mathfrak{M}, \ \xi \in \tilde{\mathfrak{D}}(\mathfrak{M}).$$

Then $\tilde{M} \equiv \{\tilde{X}; X \in \mathfrak{M}\}$ is a closed O^*-algebra on $\tilde{\mathfrak{D}}(\mathfrak{M})$, which is the smallest closed extension of $\mathfrak{M}$, and so $\mathfrak{M}$ is said to be the closure of $\mathfrak{M}$. It is well-known[7] that $\mathfrak{M}$ is closed iff $\mathfrak{D} = \hat{\mathfrak{D}}(\mathfrak{M})$.

A vector $\xi \in \mathfrak{D}$ is said to be strongly cyclic for $\mathfrak{M}$ if $\mathfrak{M}\xi$ is dense in $\mathfrak{D}[t_{\mathfrak{M}}]$. If $\mathfrak{D}^*(\mathfrak{M}) \equiv \bigcap_{X \in \mathfrak{M}} \mathfrak{D}(X^*) = \mathfrak{D}$, then $\mathfrak{M}$ is said to be self-adjoint.

We next define a weak commutant of $\mathfrak{M}$ by

$$\mathfrak{M}'_w = \{C \in \mathfrak{B}(\mathcal{H}(\mathfrak{D})); (CX\xi|\eta) = (C\xi|X^\dagger\eta) \quad \text{for all } \xi, \eta \in \mathfrak{D} \text{ and } X \in \mathfrak{M}\}.$$

Then $\mathfrak{M}'_w$ is a weakly closed $*$-invariant subspace of $\mathfrak{B}(\mathcal{H}(\mathfrak{D}))$, but it is not necessarily an algebra[7]. If $\mathfrak{M}$ is self-adjoint, then $\mathfrak{M}'_w$ is a von Neumann algebra and $\mathfrak{M}'_w \mathfrak{D} \subset \mathfrak{D}$. There are some pathologies between invariant subspaces for $\mathfrak{M}$ and the projections. It is known by Powers[7] that the projection E'_ξ of $\mathcal{H}(\mathfrak{D})$ onto $\overline{\mathfrak{M}\xi}$ does not necessarily belong to $\mathfrak{M}'_w$, and so we need the notion of self-adjoint vectors[4]. A vector $\xi \in \mathfrak{D}$ is said to be self-adjoint for $\mathfrak{M}$ if the closure of an O^*-algebra $\mathfrak{M} \lceil \mathfrak{M}\xi$ is self-adjoint. If ξ is a self-adjoint vector for $\mathfrak{M}$, then $E'_\xi \in \mathfrak{M}'_w$ and $E'_\xi\mathfrak{D} = \tilde{\mathfrak{D}}(\mathfrak{M} \lceil \mathfrak{M}\xi) \subset \mathfrak{D}$[7]. We have obtained the result[4] that $\mathfrak{M}$ is decomposed into a direct sum $\mathfrak{M}_1 \oplus \mathfrak{M}_2$ of a direct sum $\mathfrak{M}_1$ of self-adjoint O^*-algebras with strongly cyclic vector and a closed O^*-algebra $\mathfrak{M}_2$ which does not admit any non-zero self-adjoint vector. We remark that there exist self-adjoint O^*-algebras which do not admit any non-zero self-adjoint vector. An O^*-algebra $\mathfrak{M}$ is said to be regular if $\mathfrak{M} = \mathfrak{M}_1$, that is, there exists a family $\{\eta_\lambda\}$ of self-adjoint vectors for $\mathfrak{M}$ such that $\{E'_{\eta_\lambda}\}$ is mutually orthogonal and $\sum_\lambda E'_{\eta_\lambda} = 1$, and $\{\eta_\lambda\}$ is said to be a regular basis for $\mathfrak{M}$.

A σ-weak topology on $\mathfrak{M}$ is defined by a family $\{P_{\{\xi_n\},\{\eta_n\}}(\cdot); \{\xi_n\}, \{\eta_n\} \in \mathfrak{D}^\infty(\mathfrak{M})\}$ of seminorms :

$$P_{\{\xi_n\},\{\eta_n\}}(X) = \left| \sum_{n=1}^{\infty} (X\xi_n \mid \eta_n)\right|, \ X \in \mathfrak{M}$$

where

$$\mathfrak{D}^\infty(\mathfrak{M}) = \left\{\{\xi_n\} \subset \mathfrak{D}; \ \sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty \text{ and } \sum_{n=1}^{\infty} \|X\xi_n\|^2 < \infty \text{ for all } X \in \mathfrak{M}\right\}$$

Let φ be a linear functional on $\mathfrak{M}$. If $\varphi(X^\dagger X) \geq 0$ for all $X \in \mathfrak{M}$, then φ is called positive. If $\varphi(X) \geq 0$ for all $X \in \mathfrak{M}_+ \equiv \{X \in \mathfrak{M}; (X\xi|\xi) \geq 0 \text{ for all } \xi \in \mathfrak{D}\}$, then φ is called strongly positive. A positive linear functional φ on $\mathfrak{M}$ is said to be a

σ-vector form if there exists an element $\{\xi_n\}$ of $\mathfrak{D}^\infty(\mathfrak{M})$ such that

$$\varphi(X) = \sum_{n=1}^{\infty} \omega_{\xi_n}(X) \equiv \sum_{n=1}^{\infty} (X\xi_n|\xi_n)$$

for all $X \in \mathfrak{M}$.

We finally review an ampliation of an O^*-algebra $\mathfrak{M}$, an induction of $\mathfrak{M}$ and a spatial isomorphism of $\mathfrak{M}$ onto an O^*-algebra $\mathfrak{N}$.
Let $\mathcal{K}$ be a Hilbert space and put

$$\mathfrak{D} \tilde{\otimes} \mathcal{K} = \bigcap_{X \in \mathfrak{M}} \mathfrak{D}(\overline{X \otimes 1})$$

$$X \tilde{\otimes} 1 = \overline{X \otimes 1} \lceil \mathfrak{D} \tilde{\otimes} \mathcal{K}, \quad X \in \mathfrak{M}.$$

Then $\mathfrak{M} \tilde{\otimes} 1$ is a closed O^*-algebra on $\mathfrak{D} \tilde{\otimes} \mathcal{K}$ in the Hilbert space $\mathcal{H}(\mathfrak{D}) \tilde{\otimes} \mathcal{K}$. The isomorphism : $X \in \mathfrak{M} \rightarrow X \tilde{\otimes} 1 \in \mathfrak{M} \tilde{\otimes} 1$ is said to be an ampliation of $\mathfrak{M}$. Suppose $\mathfrak{M}$ is self-adjoint and E' is a projection in $\mathfrak{M}'_w$. *Then* $\mathfrak{M}_{E'} \equiv \{XE' : X \in \mathfrak{M}\}$ is a self-adjoint O^*-algebra on $E'\mathfrak{D}$. The *-homomorphism : $X \in \mathfrak{M} \rightarrow XE' \in \mathfrak{M}_{E'}$ is said to be an induction of $\mathfrak{M}$. A *-isomorphism Φ of an O^*-algebra $\mathfrak{M}$ on $\mathfrak{D}$ onto an O^*-algebra $\mathfrak{N}$ on $\mathcal{E}$ is called spatial if there exists a unitary transform U of $\mathcal{H}(\mathfrak{D})$ onto $\mathcal{H}(\mathcal{E})$ such that $U\mathfrak{D} = \mathcal{E}$ and $\Phi(X) = UXU^*$ for all $X \in \mathfrak{M}$.

3. A Decomposition of *-Homomorphisms

In this section we consider when a *-homomorphism of O^*-algebras is composed of an ampliation, an induction and a spatial isomorphism.

Lemma 3.1 – Let $\mathfrak{M}$ be a closed O^*-algebra on $\mathfrak{D}$, $\mathfrak{N}$ a self-adjoint O^*-algebra on $\mathcal{E}$ with a strongly cyclic vector η_0 and Φ a *-homomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$. Suppose that $\omega_{\eta_0} \circ \Phi$ is a σ-vector form on $\mathfrak{M}$. Then Φ is composed of an ampliation Φ_1, an induction Φ_2 and a spatial isomorphism Φ_3.

Proof : Since $\omega_{\eta_0} \circ \Phi$ is a σ-vector form on $\mathfrak{M}$, there exists an element $\{\xi_n\}$ of $\mathfrak{D}^\infty(\mathfrak{M})$ such that

$$\omega_{\eta_0} \circ \Phi = \sum_{n=1}^{\infty} \omega_{\xi_n}.$$

Let $\mathcal{K}$ be a separable Hilbert space. We put

$$\Phi_1(X) = X \tilde{\otimes} 1, \quad X \in \mathfrak{M}$$

$$\xi = \{\xi_n\} \in \mathfrak{D} \tilde{\otimes} \mathcal{K}.$$

Then, we have

$$(\Phi(X)\, \eta_0|\eta_0) = (\Phi_1(X)\, \xi|\xi), \quad X \in \mathfrak{M}. \qquad \ldots(3.1)$$

Furthermore, $\xi \in \mathfrak{D} \tilde{\otimes} \mathcal{K}$ is a self-adjoint vector for the O^*-algebra $\Phi_1(\mathfrak{M})$.
In fact, take an arbitrary $\eta \in \mathfrak{D}^*(\Phi_1(\mathfrak{M}) \lceil \Phi_1(\mathfrak{M})\xi)$.
By (3.1), the map:

$$\Phi(X)\eta_0 \rightarrow \Phi_1(X)\xi, \; X \in \mathfrak{M}$$

is extended to the unitary transform V of $\mathcal{H}(\mathcal{E})$ onto $\overline{\Phi_1(\mathfrak{M})\xi}$, and

$$\begin{aligned}|(\Phi(X)\Phi(Y)\eta_0|V^*\eta)| &= |(\Phi_1(X)\Phi_1(Y)\xi|\eta)|\\ &= |(\Phi_1(Y)\xi|(\Phi_1(X)\lceil\Phi_1(M)\xi)^*\eta|\\ &\leq \|(\Phi_1(X)\lceil\Phi_1(\mathfrak{M})\xi)^*\eta\|\ \|\Phi(Y)\eta_0\|\end{aligned}$$

for all $X, Y \in \mathfrak{M}$, and so it follows since η_0 is strongly cyclic for $\mathfrak{N} = \Phi(\mathfrak{M})$ that $V^*\eta \in \mathfrak{D}(\Phi(X)^*)$. Hence, we have

$$V^*\eta \in \bigcap_{X \in \mathfrak{M}} \mathfrak{D}(\Phi(X)^*) = \mathfrak{D}^*(\mathfrak{N}) = \mathcal{E}.$$

Since η_0 is a strongly cyclic vector for $\mathfrak{N}$, there exists a net $\{X_\alpha\}$ in $\mathfrak{M}$ such that

$$\lim_\alpha \Phi(X_\alpha)\eta_0 = V^*\eta$$

$$\lim_\alpha \Phi(X)\ \Phi(X_\alpha)\eta_0 = \Phi(X)V^*\eta$$

for each $X \in \mathfrak{M}$, and then

$$\lim_\alpha \Phi_1\ (X_\alpha)\xi = \eta$$

$$\lim_\alpha \Phi_1\ (X)\Phi_1(X_\alpha)\xi = V\Phi(X)V^*\eta$$

for each $X \in \mathfrak{M}$. Hence $\eta \in \tilde{\mathfrak{D}}(\Phi_1(\mathfrak{M})\lceil\Phi_1(\mathfrak{M})\xi)$. Since ξ is a self-adjoint vector for $\Phi_1(\mathfrak{M})$, it follows that $E' \in \Phi_1\ (\mathfrak{M})'_w$ and $\overline{E'\mathfrak{D}\otimes\mathcal{K}} = \hat{\mathfrak{D}}\Phi_1\ (\mathfrak{M})\lceil\Phi_1(\mathfrak{M})\xi)$, where E' is a projection of $\mathcal{H}(\mathfrak{D})\ \hat{\otimes}\ \mathcal{K}$ onto $\overline{\Phi_1(\mathfrak{M})\xi}$. We now put

$$\Phi_2(\Phi_1(X)) = (\Phi_1(X))_{E'}, \quad X \in \mathfrak{M}.$$

Then Φ_2 is an induction of the O^*-algebra $\Phi_1\ (\mathfrak{M})$ and

$$(\Phi(X)\eta_0 \mid \eta_0) = (\Phi_1(X)\xi|\xi) = ((\Phi_2 \circ \Phi_1)\ \xi|\xi),\ X \in \mathfrak{M}. \qquad \ldots(3.2)$$

Since η_0 is a strongly cyclic vector for $\Phi(\mathfrak{M})$ and ξ is a strongly cyclic vector for $(\Phi_2 \circ \Phi_1)\ (\mathfrak{M})$, it follows from (3.2) that there exists a unitary transform U of $\mathcal{H}(\mathcal{E})$ onto $\mathcal{H}(\mathfrak{D})\ \hat{\otimes}\ \mathcal{K}$ such that $U\mathcal{E} = E'\ (\mathfrak{D}\ \hat{\otimes}\ \mathcal{K})$ and $\Phi(X) = U^*\ (\Phi_2 \circ \Phi_1)\ (X)\ U$ for all $X \in \mathfrak{M}$. We put

$$\Phi_3((\Phi_2 \circ \Phi_1)\ (X)) = U^*(\Phi_2 \circ \Phi_1)\ (X)U, \quad X \in \mathfrak{M}.$$

Then, Φ_3 is a spatial isomorphism of the O^*-algebra $(\Phi_2 \circ \Phi_1)\ (\mathfrak{M})$ onto $\mathfrak{N}$ and $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$. This completes the proof.

Theorem 3.2 – Let $\mathfrak{M}$ be a closed O^*-algebra on $\mathfrak{D}$, $\mathfrak{N}$ a regular O^*-algebra on $\mathcal{E}$ with a regular basis $\{\eta_\lambda\}_{\lambda\in\Lambda}$ and Φ a *-homomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$. Suppose that $\omega_{\eta_\lambda} \circ \Phi$ is a σ-vector form on $\mathfrak{M}$ for each $\lambda \in \Lambda$. Then, $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$, where Φ_1 is an ampliation, Φ_2 is an induction and Φ_3 is a spatial isomorphism.

Proof : Since $\{\eta_\lambda\}$ is a regular basis for $\mathfrak{N}$, it follows that $F'_\lambda \equiv$ *Proj.* $\mathfrak{N}_{\eta_\lambda} \in \mathfrak{N}'_w$, $F'_\lambda\mathcal{E} \subset \mathcal{E}$, $\mathfrak{N}_{F'_\lambda}$ is a self-adjoint O^*-algebra on $F'_\lambda\mathcal{E}$ for each $\lambda \in \Lambda$, and $\mathfrak{N} = \bigoplus_{\lambda \in \Lambda}$ and $\mathfrak{N}_{F'_\lambda}$. We put

$$\Phi^\lambda(X) = \Phi(X)F'_\lambda,\ X \in \mathfrak{M}.$$

Then Φ^λ is a *-homomorphism of $\mathfrak{M}$ onto a self-adjoint O^*-algebra $\mathfrak{N}_{F'_\lambda}$ on $F'_\lambda\mathcal{E}$

with a strongly cyclic vector η_λ. It follows from Lemma 3.1 that $\Phi^\lambda = \Phi_3^\lambda \circ \Phi_2^\lambda \circ \Phi_1^\lambda$ for each $\lambda \in \Lambda$, that is, there exist a separable Hilbert space $\mathcal{K}_\lambda$, a projection E'_λ in $\Phi_1^\lambda(\mathfrak{M})'_w$ and a unitary transform U_λ of $F'_\lambda\, \mathcal{H}(\mathcal{E})$ onto $\mathcal{H}(\mathfrak{D}) \hat{\otimes} \mathcal{K}_\lambda$ such that $\Phi_1^\lambda\,(X) = X \otimes 1$ on $\mathfrak{D} \tilde{\otimes} \mathcal{K}_\lambda$, $\Phi_2^\lambda\,(\Phi_1^\lambda(X)) = \Phi_1^\lambda(X)_{E'_\lambda}$ and $\Phi_3^\lambda\,((\Phi_2^\lambda \circ \Phi_1^\lambda)(X)) = U_\lambda^*\,(\Phi_2^\lambda \circ \Phi_1^\lambda)(X)U_\lambda$ for all $X \in \mathfrak{M}$. We put

$$\mathcal{K} = \oplus_{\lambda\in\Lambda} \mathcal{K}_\lambda, \quad \Phi_1\,(\mathfrak{M}) = \mathfrak{M} \tilde{\otimes} 1 \text{ on } \mathfrak{D} \tilde{\otimes} \mathcal{K}$$

$$E' = (E'_\lambda)_{\lambda\in\Lambda}, \quad \Phi_2(\Phi_1(X)) = \Phi_1(X)_{E'},\ X \in \mathfrak{M}$$

$$U = (U_\lambda)_{\lambda\in\Lambda}, \quad \Phi_3((\Phi_2 \circ \Phi_1)(X)) = U^*(\Phi_2 \circ \Phi_1)\,(X)U,\ X \in \mathfrak{M}.$$

Then it is easily shown that Φ_1 is an ampliation, Φ_2 is an induction, Φ_3 is a spatial isomorphism and $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$. This completes the proof.

Proposition 3.3 – Let $\mathfrak{M}$, $\mathfrak{N}$, $\{\eta_\lambda\}_{\lambda\in\Lambda}$ and Φ be in Theorem 3.2. Suppose that $\Phi(\mathfrak{M}_+) \subset \mathfrak{N}_+$, and one of the following statements (i) and (ii) holds:

(i) There exists an element N of $\mathfrak{M}$ such that $\overline{N^{-1}}$ is a compact operator.

(ii) $\mathfrak{D}[t_\mathfrak{M}]$ is a Fréchet Montel space.

Then $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$, where Φ_1 is an ampliation, Φ_2 is an induction and Φ_3 is a spatial isomorphism.

Proof : Since $\Phi(\mathfrak{M}_+) \subset \mathfrak{N}_+$, it follows that $\omega_{\eta_\lambda} \circ \Phi$ is a strongly positive linear functional on $\mathfrak{M}$ for each $\lambda \in \Lambda$. Suppose that either (i) or (ii) holds. Then it was shown by Schmüdgen[8] that $\omega_{\eta_\lambda} \circ \Phi$ is a trace functional on $\mathfrak{M}$; that is, it is a σ-vector form on $\mathfrak{M}$. Therefore, the corollary follows from Theorem 3.2.

Corollary 3.4 – Let $\mathfrak{M}$ be a self-adjoint O^*-algebra on the Schwartz space $S(\mathbf{R})$ generated by

$$(Pf)\ (t) = -\,if'(t),$$

$$(Qf)\ (t) = tf(t),\ f \in S(\mathbf{R})$$

and $\mathfrak{N}$ a regular O^*-algebra on $\mathcal{E}$. Then every *-homorphism Φ of $\mathfrak{M}$ onto $\mathfrak{N}$ such that $\Phi(\mathfrak{M}_+) \subset \mathfrak{N}_+$ is composed of an ampliation, an induction and a spatial isomorphism. In particular, such a composition is possible for every *-homomorphism Φ of $\mathfrak{M}$ onto $\mathfrak{M}$ such that $\Phi(\mathfrak{M}_+) \subset \mathfrak{M}_+$.

Proof : It is well known[7] that $\mathfrak{M}$ is a self-adjoint O^*-algebra on $S(\mathbf{R})$ satisfying the condition (i) of Proposition 3.3. Therefore, the corollary follows from Proposition 3.3.

Proposition 3.5 – Let $\mathfrak{M}$ be a closed O^*-algebra on $\mathfrak{D}$ and $\mathfrak{N}$ a regular O^*-algebra on $\mathcal{E}$. Suppose that one of the following conditions (i) and (ii) holds :

(i) $\mathfrak{M}'_w = \mathfrak{M}'_b$ and $\mathfrak{M}'_w\,\mathfrak{D} \subset \mathfrak{D}$, where $\mathfrak{M}_b = \{X \in \mathfrak{M};\ \overline{X} \text{ is bounded}\}$.

(ii) $\mathfrak{M}$ satisfies condition (I) in the sense of Araki and Jurzak[1].

Then every σ-weakly continuous *-homomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$ is composed of an ampliation, an induction and a spatial isomorphism.

PROOF : Let $\{\eta_\lambda\}_{\lambda\in\Lambda}$ be a regular basis for $\mathfrak{N}$. Then $\omega_{\eta_\lambda} \circ \Phi$ is a σ-weakly continuous positive linear functional on $\mathfrak{M}$ for each $\lambda \in \Lambda$. Suppose that the condition (i) holds. Then it follows from Lemma 5.2 of Inoue et al.[6] that $\omega_{\xi_\lambda} \circ \Phi$ is a σ-vector form on $\mathfrak{M}$, and hence the corollary follows from Theorem 3.2. Suppose that the condition (ii) holds. Then every vector $\xi \in \mathfrak{D}$ is self-adjoint for $\mathfrak{M}$ as seen in Lemma 2.4 of Bhatt[2], and so we can show in similar to the proof of Lemma 5.2 of Inoue et al.[6] that $\omega_{\eta_\lambda} \circ \Phi$ is a σ-vector form on $\mathfrak{M}$ for each $\lambda \in \Lambda$. Therefore the corollay follows from Theorem 3.2.

We remark that every EW^*-algebra $\mathfrak{M}$ is a regular O^*-algebra such that $\mathfrak{M}'_w = \mathfrak{M}'_b$ and $\mathfrak{M}'_w \mathfrak{D} \subset \mathfrak{D}$, and every closed O^*-algebra satisfying condition (I) is a regular O^*-algebra. Therefore, Corollary 3.5 implies the following results:

Corollary 3.6 [Theorem 5.5 of Inoue[5]] – Every σ-weakly continuous *-homomorphism of a closed EW^*-algebra $\mathfrak{M}$ onto a closed EW^*-algebra $\mathfrak{N}$ is composed of an ampliation, an induction and a spatial isomorphism.

Corollary 3.7 [Theorem of Bhatt[2]] – Every σ-weakly continuous *-homomorphism of a closed O^*-algebra $\mathfrak{M}$ satisfying condition (I) onto a closed O^*-algebra $\mathfrak{N}$ satisfying condition (I) is composed of an ampliation, an induction and a spatial isomorphism.

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