SOME NEW CLASSES IN THE STABLE HOMOTOPY GROUPS OF THE THOM SPACE MSTOP OF STABLE UNIVERSAL TOPOLOGICAL BUNDLE

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A new family of non-zero elements in the stable homotopy groups of MSTOP is determined with the help of the spectral sequences of Adams and Bockstein. This family corresponds in cohomology to exotic characteristic classes associated with tertiary cohomology operations.

1. INTRODUCTION

In this paper we apply higher cohomology operations to the study of MSTOP—the Thom space of stable universal topological microbundles. This study has been an area of active work during the last two decades.

Denoting by BSTOP the stable classifying space of orientable topological microbundles, a result of Sullivan gives the following decomposition when localized at an odd prime \( p \)

\[
\text{BSTOP}_{(p)} = \text{BSO}_{(p)} \times B \text{Coker } J_p
\]

where \( \text{BSO}_{(p)} \) is the stable classifying space for vector bundles localized at \( p \) with \( H \)-space structure induced by tensor product of bundles. This decomposition passes to Thom space level and we have the following decomposition

\[
\text{MSTOP}_{(p)} = \text{MSO}_p \wedge M \text{Coker } J_p.
\]

The \( p \)-torsion in MSTOP, \( p \) being an odd prime, was investigated by Peterson in dimensions < 39 (for \( p = 3 \)). In fact, he studied MSPL, the stable Thom space of PL-microbundles but we know that MSTOP and MSPL are same away from 2. Mann and Milgram computed 3-torsion up to dimension 52. Both Peterson and Mann and Milgram used secondary cohomology operations. They showed that the first (lowest dimensional) secondary characteristic class \( \epsilon_2 \) given by (see Cohen et al., p. 167)

\[
\epsilon_2 = (\sigma_\bullet (\beta Q^{p-1} \beta Q^1 [1] \ast [1 - p^2]))^*, (( )^* denotes dual)
\]

defined by the secondary cohomology operation \( \Phi \) corresponding to the relation \( P^{p-1} \) \( P^1 = 0 \) gives 3-torsion. Mann and Milgram also showed that the divided powers of secondary classes do not give torsion. These results are also given in Madsen and Milgram (p. 253).
In this paper, we push this study further by taking up the first tertiary characteristic class $\epsilon_3$, given by $\epsilon_3 = (\sigma_\ast (\beta Q^0 Q^1 [1] \ast [1 - p^3]))^*$ and its divided powers. $\epsilon_3$ is defined by the tertiary cohomology operation $\Phi_3$ based on the relation $p^0 \Phi_2 = 0$ where $\Phi_2$ is the secondary cohomology operation dual to the homology operation $Q^0 Q^1$

We prove the following theorem in section 3.

**Theorem** – The tertiary characteristic class $\epsilon_3$ and its divided powers $\gamma_p^l (\epsilon_3)$ survive in the Adams spectral sequence, and give a new family of elements in $p^* \pi_\ast$ (MSTOP).

Our main tool in proving the theorem is the Adams spectral sequence (ASS). We compute the differentials in the ASS with the help of the Bockstein Spectral Sequence (BSS) which is closely related to it. This relation is described in section 2. Computations based on the above theorem for $p = 3$ up to dimension 4212 are made and shown in Table I.

2. **Adams Spectral Sequence, Bockstein Spectral Sequence and their Relationship**

In this section we use an important result of May and Milgram\(^5\) which permits us to detect differentials in the Adams spectral sequence in terms of differentials in the Bockstein spectral sequence.

For the sake of completeness, we recall the Adams spectral sequence. For any space $X$, let $[Y, X]$ denote the space of homotopy classes of maps from $Y$ to $X$ where $Y$ is a finite-dimensional space. Under certain stability conditions, we define a filtration on $[Y, X]$ as follows

Let $F^s = F^s [Y, X] = \text{Im} \{ [Y, X] \to [Y, X] \}$

under the composite projection $X_s \to X$ and $F^0 = [Y, X]$. For every $s$, we get a decreasing filtration

$F^{s+1}_s \subset F^s \subset F^{s-1} \ldots \ldots$

and we define

$F^\infty [Y, X] = \bigcap_{s = 0}^{\infty} F^s [Y, X].$

Also, we define the stable group $\{ Y, X \}$ to be the limit (under suspension) of the group of homotopy classes of maps from $S^m Y$ to $S^m X$.

**Lemma 2.1** – The decreasing filtration $\{ F^s \}$ of $\{ Y, X \}$ gives a spectral sequence $\{ E_{r}, d_r \}$ with the following properties:

1. $d_r$ is a homomorphism of bi-graded groups,

$d_r^{s,l} : E_r^{s,l} \to E_r^{s+r,l+r-1}$, where $d_r d_r = 0$

2. $E_2^{s,l} \simeq \text{Ext}^{s,l}_A \{ H^*(X; \mathbb{Z}/p); H^*(Y; \mathbb{Z}/p) \}$
(3) \( E_{\infty}^{s,t} = \bigcap_{r > s} E_r^{s,t} \)

(4) \( E_{\infty}^{s,t} = F^s \{ S^{t-s} Y, X \}/F^{s+1} \{ S^{t-s} Y, X \} \) \hfill \Box

In the next section, we apply the above theorem to the case \( X = \text{MSTOP} \) and \( Y = S^0 \).

The following result due to Liulevicius\(^2\) establishes a vanishing line in the \( E_2 \)-term of the ASS. Let \( A \) be the Steenrod algebra modulo an odd prime \( p \) and \( A_0 = E(\beta) \subset A \) where \( \beta \) denotes cohomology Bockstein.

**Lemma 2.2** – Let \( M \) be an \((m - 1)\)-connected \( A_0 \)-free \( A \)-module. Then, \( \text{Ext}_A^{s,t}(M, Z/p) = 0 \) for \( s \geq 1 \) and \( t - s \leq m + f(s) \) where \( f(s) = 2(p - 1)s \).

We also recall here that in the homology BSS modulo \( p \) of a spectrum \( X \), \( E^i X = H_\ast(X; Z/p) \) and the differential \( d' \) is the \( r \)-th homology Bockstein operation \( \beta_r \) which is derived from the short exact sequence

\[
0 \rightarrow Z/p \rightarrow Z/p^{r+1} \rightarrow Z/p^r \rightarrow 0.
\]

We now state the result of May and Milgram which connects the mod \( p \) homology BSS \{\( E^i X \}\} to the ASS \{\( E_r X \}\} which converges to \( p\pi_\ast(X) \). We mention here that \( E_r X \) is a left \( E_r_S \) module where \( S \) is the sphere spectrum and \( a_0 \in E_r^{1,1} S \) is an infinite cycle which amounts to multiplication by \( p \) in the \( E_\infty \)-term.

If \( x \in \text{Ext}_A(M, Z/p) \), \( M \) being an \( A \)-module, is not of the form \( a_0 x' \) for some \( x' \in \text{Ext}_A(M, Z/p) \) and if \( a_0 x \neq 0 \) for all \( i \), then we say that \( x \) generates a "spike".

We now state the basic result we need.

**Lemma 2.3** – Let \( X \) be an \((m - 1)\)-connected spectrum with integral homology of finite type.

(i) The set of spikes in the \( r \)-th term \( E_r X \) of the ASS \((2 \leq r \leq \infty)\) is in one-to-one correspondence with a basis of the \( r \)-th term \( E^r X \) of the homology BSS mod \( p \) of \( X \). If a basis element has degree \( q \) and the corresponding spike is generated by \( \alpha \in E_r^{s,t} X \) then \( f(s) + m \leq q = t - s \).

(ii) If \( \delta \in E_r^{s,t} X \) and \( \epsilon \in E_u^{u,v} X \), \( v - u = t - s - 1 \), generate the spikes corresponding to basis elements \( d \) and \( \beta_r d \). Then

\[
d_r(a_0^i \delta) = a_0^{i+r+s-u} \epsilon
\]

provided that \( m + f(i + s) \geq t - s \). \hfill \Box

3. An Infinite Family in \( p\Omega_{\ast}^{\text{TOP}} \)

We now take up the Adams spectral sequence whose \( E_2 \)-term is

\[
E_2^{s,t} = \text{Ext}_A^{s,t}(H^\ast(\text{MSTOP}; Z/p); Z/p)
\]

which converges to the \( p \)-component of \( \pi_\ast(\text{MSTOP}) \) i.e., \( p\pi_\ast^{(\text{MSTOP})} \).
We recall from Milnor\(^6\) the elements \(Q_i\) of \(A\),

\[
Q_0 = \beta, \quad Q_{i+1} = Q_i \cdot P^p - P^p \cdot Q_i.
\]

They are odd dimensional generators and generate an exterior subalgebra \(E(Q_0, Q_1, \ldots, Q_i, \ldots)\) of \(A\). The following proposition considerably simplifies the computation of \(E_2\)-term of the Adams spectral sequence for \(\text{MSTOP}\).

**Lemma 3.1** — The computations of \(\text{ASS}\) for \(H^*\) (\(\text{MSTOP}\)) as module over \(A\) reduce to computation of \(H^*\) \((M \text{Coker } J_p)\) as module over \(E(Q_0, Q_1, \ldots)\).

**Proof:** Since,

\[
\text{Ext}_A [H^*_{\text{MSTOP}}(p), Z/p] = \text{Ext}_A [H^*(\text{MSO}_p) \otimes H^*(M \text{Coker } J_p), Z/p]
\]

and (see Peterson\(^7\))

\[
H^*(\text{MSO}_p) \equiv \text{a direct sum of copies of } A/\langle E(Q_0, Q_1, \ldots) \rangle.\text{ Therefore, by change of rings theorem,}
\]

\[
\text{Ext}_A [A/\langle E(Q_0, Q_1, \ldots) \rangle \otimes H^*(M \text{ Coker } J_p), Z/p] = \text{Ext}_{E(Q_0, Q_1, \ldots)} [H^* [M \text{ Coker } J_p]; Z/p]
\]

since for any \(N\), the \(A\)-module structure of \(A/\langle E(Q_0, Q_1, \ldots) \rangle \otimes N\) depends only on the \(E(Q_0, Q_1, \ldots)\)-module structure of \(N\). Hence the \(A\)-action on \(H^*\) \((\text{MSTOP}_p)\) boils down to the \(E(Q_0, Q_1, \ldots)\)-action on a direct sum of copies of \(H^*(M \text{Coker } J_p)\). So, we need consider only \(H^*(M \text{Coker } J_p)\) as \(E(Q_0, Q_1, \ldots)\)-module. Now since any generator of \(H^*(B \text{Coker } J_p)\) can be lifted, by taking cup product with the Thom class \(U\), to a generator of \(H^*(M \text{Coker } J_p)\), we will suppress this lifting and take for generators of \(H^*(M \text{Coker } J_p)\) the corresponding generators of \(H^*(B \text{Coker } J_p)\). Knowledge of \(H_*(B \text{Coker } J_p, Z/p)\) \((\text{Cohen et al.}^1, \text{p. 167})\) gives (see Singh\(^8\), \text{p. 65})

\[
H^*(B \text{Coker } J_p ; Z/p) = E \{ \epsilon^j_i \} \otimes_A \Gamma_p \{ \epsilon^j_i \} \text{ as Hopf algebra over } A, \text{ where } l \text{ varies in the set of admissible sequence and } i \geq 2, j \geq 2. \text{E and } \Gamma_p \text{ denote exterior and divided power algebras.}
\]

Recall that since all differentials vanish, \(E_2 \text{ MSO} = E_{\infty} \text{ MSO}\) and \(a_i \in E_2^{1,2^i-1}, \text{MSO generate the polynomial algebra } P \{ a_i \mid i \geq 0 \} \subset E_2 \text{ MSO.}
\]

At this stage, we must bring in the BSS of \(\text{MSTOP}\) which is given by

\[
E' \text{ MSTOP} = E' M \text{ Coker } J_p \otimes E' \text{ MSO} = E' B \text{ Coker } J_p \otimes H_* \text{ MSO}.
\]

The BSS of \(B \text{ Coker } J_p\) is given by the following result due to May.

**Lemma 3.2** \((\text{Cohen et al.}^1, \text{p. 170})\) For all \(r \geq 1\), the homology Bockstein spectral sequence of \(B \text{Coker } J_p\) is given by

\[
E^{r+1} = P[y^p] \otimes [y^{p-1} \beta y] \text{ with } \beta_{r+1} y^p = y^{p-1} \beta y
\]

where \(y\) runs through \(\{ \sigma_* y_I | \epsilon_1 = 0, d(I) \text{ odd} \}\), where \(\sigma_* y_I\) are in the generating set of \(H_* (B \text{Coker } J_p)\) \((\text{Cohen et al.}^1)\). \(\Box\)
For $p = 3$, in dimensions $\leq 52$, $E^2B \text{Coker } J_3 = P \{ \gamma_3 \left( Q_0 \epsilon_2 \right) \} \otimes E \{ \epsilon_2 \left( Q_0 \epsilon_2 \right)^2 \}$ with $\beta_2 \left( \gamma_3 \left( Q_0 \epsilon_2 \right) \right) = \epsilon_2 \left( Q_0 \epsilon_2 \right)^2$.

The $E_2$-term of the Adams spectral sequence is due to Mann and Milgram. We have added the first tertiary class $\epsilon_3$ which lies in dimension 52 for $p = 3$.

The elements $a_i^j$ etc. denote only the leading term and $a_2 = -[a_0 \left( Q_1 \epsilon_2 \right) + a_1 \left( Q_0 \epsilon_2 \right)]$.

In this range the following non-trivial differentials occur and are known.

**Lemma 3.3**

1. $d_2 \left( a_0^i \gamma_3 \left( Q_0 \epsilon_2 \right) \right) = a_0^{i+2} \epsilon_2 \left( Q_0 \epsilon_2 \right)^2, i \geq 0$
2. $d_2 \left( a_0^i a_1 \gamma_3 \left( Q_0 \epsilon_2 \right) \right) = a_0^{i+3} \epsilon_2 Q_0 \left( \epsilon_2 \right) Q_1 \left( \epsilon_2 \right), i \geq 0$
3. $d_2 \left( a_0^i a_1^j \gamma_3 \left( Q_0 \epsilon_2 \right) \right) = a_0^{i+4} a_j^i \epsilon_2 \left( Q_1 \epsilon_2 \right)^2, i \geq 0, j = 0, 1, 2.$
4. $d_2 \left( a_0^i a_1^j \gamma_3 \left( Q_1 \epsilon_2 \right) \right) = a_0^{i+1} a_1^{i+1} \epsilon_2 \left( Q_1 \epsilon_2 \right)^2, i \geq 0, j = 0, 1$
5. $d_2 \left( a_0^i a_2 \gamma_3 \left( Q_0 \epsilon_2 \right) \right) = a_0^{i+3} \epsilon_2 \left( Q_0 \epsilon_2 \right)^2 Q_1 \left( \epsilon_2 \right), i \geq 0$
6. $d_2 \left( W \right) = a_1^2 \epsilon_2 \left( Q_1 \epsilon_2 \right)^2$.

**Proof**: In the region above the vanishing line, given by $s \geq 2$ and $t-s < 2(p-1)s$, the differential $d_2$ comes from the BSS differential $\beta_2$. Recall that $\beta_2 \left( \gamma_3 \left( Q_0 \epsilon_3 \right) \right) = \epsilon_2 \left( Q_0 \epsilon_2 \right)^2$. These differentials also hold true below the vanishing line by dividing by $a_0$ since $a_0$ is an $\infty$-cycle and each spike is uniquely generated. (2) and (3) follow from (1) by using relations between $\epsilon_2 \left( Q_1 \epsilon_2 \right)$ and $\epsilon_2 \left( Q_0 \epsilon_2 \right)$. Now, $a_2 = -\left( a_0 \left( Q_1 \epsilon_2 \right) + a_1 \left( Q_0 \epsilon_2 \right) \right)$ implies $-a_2^3 = a_0^3 \left( Q_1 \epsilon_2 \right)^3 + a_1^3 \left( Q_0 \epsilon_2 \right)^3$. The fact that $a_3^2$ is an $\infty$-cycle, establishes (4). (5) follows from (1) and (6) is a consequence of (1) and (4), since $Q_0 \left( W \right) = Q_2 \left( \gamma_p \left( Q_0 \epsilon_2 \right) \right)$ where $\gamma_p$ is the divided power. □

**Proof of the Main Theorem**

We compute differentials $d_r$ on the first tertiary class $\epsilon_3$ and its divided powers. Obviously, $\epsilon_3$ cannot be hit from below. For $r = 2$, the possible elements which can be hit through $d_2$ are the following (see $E_2$-term in dimension 51)

(i) $a_0^i a_1^2 \epsilon_2 \left( Q_1 \epsilon_2 \right)^2, i \geq 0$

(ii) $a_0^{i+2} \epsilon_2 \left( Q_0 \epsilon_2 \right)^2 Q_1 \epsilon_2, i \geq 0$

Now, by virtue of differentials (4), (5) and (6) in Lemma 3.3 all elements in (i) and (ii) except $a_0^3 \epsilon_2 \left( Q_0 \epsilon_2 \right)^2 Q_1 \epsilon_2$ are hit through $d_2$ by elements other than $\epsilon_3$. We claim that $\epsilon_3$ does not hit either $a_0^3 \epsilon_2 \left( Q_0 \epsilon_2 \right)^2 Q_1 \epsilon_2$ or $a_1^2 \epsilon_2 \left( Q_1 \epsilon_2 \right)^2$, which will imply that $d_2 \left( \epsilon_3 \right) = 0$.

Suppose that $d_2 \left( \epsilon_3 \right) = a_0^3 \epsilon_2 \left( Q_0 \epsilon_2 \right)^2 Q_1 \epsilon_2$. Since the $E_2$-term is a left $E_2S$-module we multiply by $a_0 \in E_2^{1,1} S$ to get

$a_0 d_2 \left( \epsilon_3 \right) = a_0^3 \epsilon_2 \left( Q_0 \epsilon_2 \right)^2 Q_1 \epsilon_2$. 

Now, using the property that for any \( u \in E_1^{\text{t-s}} S \) and \( v \in E_1 X \),
\[
d_1 (uv) = d_1 (u) v + (-1)^{t-s} ud_1 (v)
\]
we have:
\[
d_2 (a_0 \varepsilon_3) = d_2 (a_0) \varepsilon_3 + a_0 d_2 (\varepsilon_3)
= a_0 d_2 (\varepsilon_3), \text{ since } a_0 \text{ an } \infty\text{-cycle.}
\]

We can not write
\[
d_2 (a_0 \varepsilon_3) = a_0^3 \varepsilon_2 (Q_0 \varepsilon_2)^2 Q_1 \varepsilon_2.
\]

We recall that
\[
\varepsilon_3 = [\sigma_* (\beta Q^{p_1} Q^p Q^1 [1] [1 - p^3])]^*.
\]

Hence, by Lemma 3.2, \( \varepsilon_3 \) does not fall in one-to-one correspondence with any basis element of the BSS \( E^2 B \) Coker \( J_p \) and also of \( E^2 \text{MSTOP} \) and thus by (i) of Lemma 2.3, \( \varepsilon_3 \) will not generate a spike. This implies that \( a_0 \varepsilon_3 = 0 \), which further implies that \( a_0 \varepsilon_3 = 0 \), which further implies that \( d_2 (a_0 \varepsilon_3) = 0 \) giving \( a_0^3 \varepsilon_3 (Q_0 \varepsilon_2)^2 Q_1 \varepsilon_2 = 0 \) which is a contradiction since \( a_0^3 \varepsilon_2 (Q_0 \varepsilon_2)^2 Q_1 \varepsilon_2 \neq 0 \) (see \( E_2 \)-term in dimension 51). Hence, \( d_2 (\varepsilon_3) \neq a_0^3 \varepsilon_2 (Q_0 \varepsilon_2)^2 Q_1 \varepsilon_2 \). A similar argument proves that \( d_2

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**Fig. 1.** The \( E_2 \)-term of the Adams spectral sequence for \( p=3, 51 \leq t-s \leq 53 \). (Some \( Z/3 \) generators in filtration zero in dimensions 51 and 52 have been ignored.)
(ε₃) ≠ a₁²ε₂(Q₁ε₂)². Since there is no other element which can be hit by ε₃ under d₂, we conclude that d₂(ε₃) = 0. Since all elements except a₂²ε₂(Q₀ε₂)²Q₁ε₂ are being hit through d₂ by elements in dimension 52, there is no possibility of d₃(ε₃) being non-zero. This is true for all r > 2, implying that d_r(ε₃) = 0, ∀r > 2. Thus d_r(ε₃) = 0 ∀r ≥ 2, hence ε₃ is an ∞-cycle in the ASS and it will generate a Z/p summand. This summand cannot be Z/pᵏ, k > 1, because ε₃ does not generate a spike i.e. a₀ε₃ = 0.

Let us denote by γ_p^(j) (ε₃) the iterated divided powers of ε₃ where γ_p (ε₃) = \frac{ε₃}{p!} and γ_p^(j) = γ_p ... γ_p (j times).

Now, since d_r r ≥ 2 is a derivation and d_r(ε₃) = 0, we immediately get d_r(γ_p^(j) (ε₃)) = 0, thus implying that γ_p^(j) (ε₃) are ∞-cycles in the ASS.

Let Ωⁿ_TOp be the n-dimensional cobordism group of topological manifolds. It is known that Ωⁿ_TOp ≅ πₙ (MSTOP). Using the above computations in the ASS we obtain the 3-torsion in Ωⁿ_TOp in Table I.

**TABLE I**

**Some 3-torsion in Ωⁿ_TOp contributed by tertiary classes**

<table>
<thead>
<tr>
<th>n</th>
<th>eΩⁿ_TOp</th>
</tr>
</thead>
<tbody>
<tr>
<td>52</td>
<td>Z/3</td>
</tr>
<tr>
<td>156</td>
<td>Z/3'r</td>
</tr>
<tr>
<td>468</td>
<td>Z/3's</td>
</tr>
<tr>
<td>1404</td>
<td>Z/3't</td>
</tr>
<tr>
<td>4212</td>
<td>Z/3'u</td>
</tr>
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Here r, s, t, u are positive integers ≥ 1

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