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BENDING OF SYMMETRICALLY LOADED CIRCULAR PLATE OF VARIABLE THICKNESS

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Solution has been obtained for the problem of a uniformly compressed and symmetrically loaded circular annular plate of varying thickness. Deflection in the plate has been obtained and compared with the corresponding deflection in a plate with uniform thickness.

1. INTRODUCTION

The problem of bending of circular plate of variable thickness was first investigated by Holzer (1918). The symmetrical bending of circular plates was considered by Pichler (1928), Olsson (1937) and more recently by Conway (1948), Fung (1953) and Mansfield (1962). In his later work Conway (1951) solved the problem of axially symmetrical plates with linearly varying thickness.

In the present investigation, the differential equation of equilibrium has been obtained for the bending of a normally loaded and uniformly compressed circular plate with a central hole and variable thickness. Solution has been obtained for the case in which the thickness h of the plate varies as an arbitrary power m of the radial distance from the centre of the plate, as given by

$$h = h_0 \left(\frac{r}{a}\right)^m \qquad \ldots(1)$$

where h_0 is the value of h on the outer boundary $r = a$ of the plate and m is a rational number. Solutions have been obtained of the governing differential equation under the following loading conditions:

(i) A load P uniformly distributed around the periphery of the hole,

(ii) uniform loading over the whole plate.

These two loading conditions will be referred to as Case I and Case II respectively in our latter discussion.

Expressions for deflection have been obtained for two different boundary conditions for each of the two types of loading given above and have been compared with the deflections obtained in the associated problem with uniform thickness.

2. Governing Equation and Solution

With the assumption of small deflection theory of plates, we have, for the equilibrium of an element of plate of thickness h bounded by two cylindrical surfaces of radii r and $r + dr$ and two radial surfaces making an angle $d\theta$ with each other (Timoshenko and Woinowsky-Krieger 1959)

$$\left[M_r + \frac{dM_r}{dr}\right](r + dr)\, d\theta - M_r r\, d\theta - M_t\, dr\, d\theta + (Q + Th\phi)\, r\, dr\, d\theta = 0 \qquad \ldots(2)$$

where $\phi = -dw/dr$ and w are slope and deflection of the middle surface of the plate respectively. Q is the shearing force per unit length acting normally to the middle surface, T stands for uniform pressure per unit length and thickness, M_r and M_t are the bending moments per unit length of section perpendicular to the radius and tangent respectively, as given by

$$M_r = D\left(\frac{d\phi}{dr} + \nu\frac{\phi}{r}\right),\quad M_t = D\left(\frac{\phi}{r} + \nu\frac{d\phi}{dr}\right). \qquad \ldots(3)$$

D in (3) is the flexural rigidity of the plate given by

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad \ldots(4)$$

where E and ν are the constant Young's modulus and Poisson's ratio respectively.

Neglecting the second order quantities (3) becomes

$$M_r + r\left[\frac{dM_r}{dr}\right] - M_t + r(\phi + Th\phi) = 0$$

which in view of (1), (3) and (4) yields the governing equation as

$$\frac{d^2\phi}{dr^2} + \frac{1+3m}{r}\frac{d\phi}{dr} + \frac{3m\nu - 1}{r^2}\phi + \frac{12(1-\nu^2)\,Ta^{2m}}{Eh_0^2}\cdot\frac{1}{r^{2m}}\phi = -\frac{12(1-\nu^2)Qa^{3m}}{Eh_0^3}\cdot\frac{1}{r^{3m}}. \qquad \ldots(5)$$

Writing

$$\phi = r^{-(3m/2)}f(z),\quad z = \beta r^{1-m} \qquad \ldots(6)$$

(5) becomes

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} + (z^2 - p^2)f = -\frac{Qa^2}{Th_0\beta^{m/2\gamma}}\cdot z^{(2-m+2\gamma)/2} \qquad \ldots(7)$$

in which

$$\gamma = 1 - m,\quad \beta^2 = \frac{12(1-\nu^2)\,Ta^{2m}}{Eh_0^2\,(1-m)^2}$$

and

$$p^2 = \frac{(3m - 2\nu)^2 + 4(1 - \nu^2)}{4(1 - m)^2} > 0, \ (m \neq 1). \qquad \text{...(8)}$$

For the case $m = 1$, the differential eqn. (5) reduces to Euler type and has been attempted by Dhaliwal (1968).

We attempt the solution of (5) under two loading conditions as below

Case I — Normal load P distributed around the periphery of the hole, such that

$$Q = \frac{P}{2\pi r}.$$

The solution of (5) in this case is (Erdelyi 1953, p. 40)

$$\phi = r^{-3m/2}\left[AJ_p(\beta r^\gamma) + BY_p(\beta r^\gamma)\right] - \frac{K\beta^{(2-3m)/2\gamma}}{((2 - 3m)/2\gamma)^2 - p^2} \cdot r^{1-3m} \cdot G(-\tfrac{1}{4}\beta^2 r^{2\gamma}) \qquad \text{...(9)}$$

where $K = \frac{Pa^m}{\pi h_0 T}\beta^{(2-m)/2\gamma}$

and $G\left(-\frac{z^2}{4}\right) = {}_1F_2\left\{1; \tfrac{1}{2}\left(3 - p - \frac{m}{2\gamma}\right), \tfrac{1}{2}\left(3 + p - \frac{m}{2\gamma}\right), -\frac{z^2}{4}\right\}.$

Case II — When there is a uniformly distributed load of intensity q, such that

$$Q = \frac{1}{2\pi r}\int_b^r 2\pi q r \, dr$$

in which b has been assumed to be the internal radius of the annular plate.

The solution of (5) in this case is (Erdelyi 1953, p. 40)

$$\phi = r^{-3m/2}\left[AJ_p(\beta r^\gamma) + BY_p(\beta r^\gamma)\right] + \frac{Lb^2\beta^{(6-3m)/2\gamma}\, r^{1-3m}}{((2 - 3m)/2\gamma)^2 - p^2} G(-\tfrac{1}{4}\beta^2 r^{2\gamma}) - \frac{L\beta^{(6-3m)/2\gamma}\, r^{3-3m}}{((6 - 3m)/2\gamma)^2 - p^2} H(-\tfrac{1}{4}\beta^2 r^{2\gamma}) \qquad \text{...(10)}$$

where $H\left(-\frac{z^2}{4}\right) = {}_1F_2\left(1; \tfrac{1}{2}\left(\frac{4 - m}{2\gamma} + 3 - p\right), \tfrac{1}{2}\left(\frac{4 - m}{2\gamma} + 3 + p\right), -\frac{z^2}{4}\right)$

and $L = \frac{qa^m}{2Th_0\beta^{(m+2)/2\gamma}}.$

In (9) and (10) J_p and Y_p are Bessel functions of first and second kind of order p and ${}_1F_2$ is the hypergeometric series defined by

$$ {}_1F_2(a;\, b, c, z) = 1 + \frac{a}{bc}\cdot\frac{z}{1!} + \frac{a(a+1)}{b(b+1)\,c(c+1)}\cdot\frac{z^2}{2!} $$

$$ + \frac{a(a+1)\,(a+2)}{b(b+1)\,(b+2)\,c(c+1)\,(c+2)}\cdot\frac{z^3}{3!} + \dots . $$

Using the formula $\phi = -\dfrac{dw}{dr}$ we have, in case I

$$ w = w_1 + \frac{1}{\gamma}\,\beta^{(m/2\gamma)-1}\,[A_1I_1(z) + B_1I_2(z) + I_3(z)] \qquad \text{...(11)} $$

and in case II

$$ w = w_2 + \frac{1}{\gamma}\,\beta^{(m/2\gamma)-1}\,[A_2I_1(z) + B_2I_2(z) + I_4(z)] \qquad \text{...(12)} $$

where w_1 and w_2 are the values of w in the two cases respectively at $z = z_0$ corresponding to the external boundary $r = a$ and

$$ I_1(z) = \int_z^{z_0} z^{-(m/2\gamma)}\, J_p(z)\, dz,\ \ I_2(z) = \int_z^{z_0} z^{-(m/2\gamma)}\, Y_p(z)\, dz $$

$$ I_3(z) = -\frac{K}{((2-3m)/2\gamma)^2 - p^2}\int_z^{z_0} z^{(1-2m)/\gamma}\, G(-\tfrac{1}{4}z^2)\, dz $$

$$ I_4(z) = \frac{Lb^2\beta^{2/\gamma}}{((2-3m)/2\gamma)^2 - p^2}\int_z^{z_0} z^{(1-2m)/\gamma}\, G(-\tfrac{1}{4}z^2)\, dz $$

$$ -\frac{L}{((6-3m)/2\gamma)^2 - p^2}\int_z^{z_0} z^{(3-2m)/\gamma}\, H(-\tfrac{1}{4}z^2)\, dz. $$

3. Solutions under Different Boundary Conditions

Two cases of practical interest are now considered.

(a) *External edge clamped and supported and the internal edge clamped* — The boundary conditions are, in this case,

$$ \phi(a) = \phi(b) = 0 $$

$$ w(a) = 0. $$

(b) *External edge simply supported and internal edge clamped* — The boundary conditions are, in this case,

$$\left(\frac{d\phi}{dr} + \nu \frac{\phi}{r}\right)_{r=a} = 0, \ \ \phi(b) = 0,$$

and

$$w(a) = 0.$$

These two boundary conditions will be referred to as (a) and (b) respectively in our latter discussions.

Next we determine the constants A_1, B_1 and A_2, B_2 of (11) and (12) for the two different loadings under the boundary conditions (a) and (b).

Case I — In this case for both (a) and (b) $w_1 = 0$ in (11). For (a)

$$A_1 = \frac{1}{\Delta_1} \{h_1(b)\, Y_p(\beta a^\gamma) - h_1(a)\, Y_p(\beta b^\gamma)\}$$

$$B_1 = \frac{1}{\Delta_1} \{h_1(a)\, J_p(\beta b^\gamma) - h_1(b)\, J_p(\beta a^\gamma)\}$$

where

$$h_1(y) = -\frac{K\beta^{(2-3m)/2\gamma}\, y^{(2-3m)/2}}{((2-3m)/2\gamma)^2 - p^2} \cdot G(-\tfrac{1}{4}\beta^2 y^{2\gamma})$$

and

$$\Delta_1 = J_p(\beta a^\gamma)\, Y_p(\beta b^\gamma) - J_p(\beta b^\gamma)\, Y_p(\beta a^\gamma).$$

For (b)

$$A_1 = \frac{1}{\Delta_2} \{h_1(b)\, g_2(a) - h_2(a) Y_p(\beta b^\gamma)\}$$

$$B_1 = \frac{1}{\Delta_2} \{h_2(a)\, J_p(\beta b^\gamma) - h_1(b)\, f_2(a)\}.$$

where

$$f_2(y) = \gamma\beta y^\gamma J_p'(\beta y^\gamma) - \left(\nu - \frac{3m}{2}\right) J_p(\beta y^\gamma)$$

$$g_2(y) = \gamma\beta y^\gamma Y_p'(\beta y^\gamma) + \left(\nu - \frac{3m}{2}\right) Y_p(\beta y^\gamma)$$

$$h_2(y) = \frac{K\beta^{(2-3m)/2\gamma}\, y^{1-(3m/2)}}{((2-3m)/2\gamma)^2 - p^2} \left\{\tfrac{1}{2}\gamma\beta^2 y^{2\gamma} G'(-\tfrac{1}{4}\beta^2 y^{2\gamma}) + (3m - 1 - \nu)\, G\left(-\frac{\beta^2 y^{2\gamma}}{4}\right)\right\}$$

and $\quad \Delta_2 = f_2(a)\, Y_p(\beta b^\gamma) - g_2(a)\, J_p(\beta b^\gamma).$

The dashes above denote differentiation of the function with respect to the argument.

Case II — In this case for both (a) and (b) $w_2 = 0$ in (12). For (a)

$$A_2 = \frac{1}{\Delta_1} \{h_3(b)\, Y_p(\beta a^\gamma) - h_3(a)\, Y_p(\beta b^\gamma)\}$$

$$B_2 = \frac{1}{\Delta_1} \{h_3(a)\, J_p(\beta a^\gamma) - h_3(b)\, J_p(\beta a^\gamma)\}$$

where

$$h_3(y) = \frac{Lb^2 \beta^{(6-3m)/2\gamma}\, y^{1-(3m/2)}}{((2-3m)/2\gamma)^2 - p^2}\, G(-\tfrac{1}{4}\beta^2 y^{2\gamma}) - \frac{L\beta^{(6-3m)/2\gamma}\, y^{3-(3m/2)}}{((6-3m)/2\gamma)^2 - p^2}\, H(-\tfrac{1}{4}\beta^2 y^{2\gamma}).$$

For (b)

$$A_2 = \frac{1}{\Delta_2} \{g_2(a)\, h_3(b) - h_4(a)\, Y_p(\beta b^\gamma)\}$$

$$B_2 = \frac{1}{\Delta_2} \{h_4(a)\, J_p(\beta b^\gamma) - h_3(b)\, f_2(a)\}$$

where

$$h_4(y) = \frac{L\beta^{(6-3m)/2\gamma} \,.\, y^{3-(3m/2)}}{\left(\frac{6-3m}{2\gamma}\right)^2 - p^2} \left\{\frac{\gamma\beta^2}{4}\, y^{2\gamma} H'\left(-\frac{\beta^2 y^{2\gamma}}{4}\right) - (3m+\nu)\, H\left(-\frac{\beta^2 y^{2\gamma}}{4}\right)\right\} - \frac{L\beta^{(6-3m)/2\gamma} \,.\, y^{1-(3m/2)}}{((2-3m)/2\gamma)^2 - p^2}$$
$$\times \left\{\frac{\gamma\beta^2}{2}\, y^{2\gamma} G'\left(-\frac{\beta^2 y^{2\gamma}}{4}\right) + (3m-\nu-1)\, G\left(-\frac{\beta^2 y^{2\gamma}}{4}\right)\right\}.$$

4. Associated Problem of Uniform Thickness

When a circular plate of uniform thickness h_0 is under simultaneous action of symmetrical lateral loading and a uniform compression T in the middle plane of the plate, the differential equation for the slope is given by, Timoshenko and Woinosky-Krieger (1959),

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \left(\frac{s^2}{a^2} - \frac{1}{r^2}\right)\phi = -\frac{Q}{D_0} \qquad \ldots(13)$$

where $\quad s^2 = Th_0 a^2/D_0,\; D_0 = \dfrac{Eh_0^3}{12(1-\nu^2)}.$

The general solution of (13) for a circular annular plate is

$$\phi(r) = A'J_1\left(\frac{sr}{a}\right) + B'Y_1\left(\frac{sr}{a}\right) + \phi_0(r) \qquad \text{...(14)}$$

where

$$\phi_0 = -\frac{P}{2\pi r}\cdot\frac{1}{r} \quad \text{or} \quad -\frac{qb}{2T}\left(\frac{r}{b} - \frac{b}{r}\right) \qquad \text{...(15)}$$

according as the normal load is uniformly distributed along the periphery of the hole or uniformly distributed over the whole plate.

For plates of uniform thickness we now write the expressions for slope and deflection for the four particular cases as discussed earlier.

Case I

(a) $$\phi = -\frac{P}{2\pi aT\Delta_3}[\{cY_1(s) - Y_1(s/c)\}\, J_1(sr/a) + \{J_1(s/c) - cJ_1(s)\}\, Y_1(sr/a) + \Delta_3\,.\,(a/r)]$$

$$w = -\frac{P}{2\pi sT\Delta_3}[\{cY_1(s) - Y_1(s/c)\}\, \{J_0(sr/a) - J_0(s)\} + \{J_1(s/c) - cJ_1(s)\}\, \{Y_0(sr/a) - Y_0(s)\} - s\Delta_3\,.\,\log(r/a)]$$

(b) $$\phi = -\frac{P}{2\pi aT\Delta_4}[\{(1-\nu)\, Y_1(s/c) + c\,[sY_1'(s) + \nu Y_1(s)]\}\, J_1(sr/a) + \{c\,[sJ_1'(s) + \nu J_1(s)] + (1-\nu)\, J_1(s/c)\}\, Y_1(sr/a) + \Delta_4\,.\,(a/r)]$$

$$w = -\frac{P}{2\pi sT\Delta_4}[\{(1-\nu)\, Y_1(s/c) + c\,[sY_1'(s) + \nu Y_1(s)]\}\, \{J_0(sr/a) - J_0(s)\} - \{c\,[sJ_1'(s) + \nu J_1(s)] + (1-\nu)\, J_1(s/c)\}\, \{Y_0(sr/a) - Y_0(s)\} - s\Delta_4\,.\,\log(r/a)].$$

Case II

(a) $$\phi = \frac{qa(c^2-1)}{2Tc^2\Delta_3}\left[Y_1(s/c)\, J_1(sr/a) - J_1(s/c)\, Y_1(sr/a) - \frac{c}{c^2-1}\Delta_3\left(\frac{r}{b} - \frac{b}{r}\right)\right]$$

$$w = \frac{qa^2(c^2-1)}{2Tc^2s\Delta_3}\left[Y_1(s/c)\, \{J_0(sr/a) - J_0(s)\} - J_1(s/c)\, \{Y_0(sr/a) - Y_0(s)\} + \frac{s\Delta_3}{2(c^2-1)}\left\{c^2\left(\frac{r^2}{a^2} - 1\right) - 2\log(r/a)\right\}\right].$$

(b) $$\phi = \frac{qa}{2T}\,\frac{(c^2+1)+\nu(c^2-1)}{c^2 . \Delta_4}\,[Y_1(s/c)\,J_1(sr/a) - J_1(s/c)\,Y_1(sr/a)] - \frac{qa}{2T}\left[\frac{r}{a} - \frac{1}{c^2}\left(\frac{a}{r}\right)\right]$$

$$w = \frac{qa^2}{2T}\,\frac{(c^2+1)+\nu(c^2-1)}{c^2 . s\Delta_4} \times [Y_1(s/c)\,\{J_0(sr/a) - J_0(s)\} - J_1(s/c)\,\{Y_0(sr/a) - Y_0(s)\}] + \frac{qa^2}{4Tc^2}\left[c^2\left(\frac{r^2}{a^2} - 1\right) - 2\log(r/a)\right]$$

where

$$\Delta_3 = J_1(s)\,Y_1(s/c) - J_1(s/c)\,Y_1(s)$$

$$\Delta_4 = [sJ_1'(s) + \nu J_1(s)]\,.\,Y_1(s/c) - [sY_1'(s) + \nu Y_1(s)]\,J_1(s/c)$$

$$c = a/b.$$

It should be noted that the parameters are to be so chosen that Δ_3 and Δ_4 as also Δ_1 and Δ_2 used in the case of non-uniform thickness are not zero, so that the deflection and slope are not infinite. To be more specific, for given m and ν, T should be so chosen that there is no buckling in the assumed finite body of the annulus. For example Δ_1, Δ_2, Δ_3 and Δ_4 will have non-zero values for $45Ta^2 = 16\,Eh_0^2$ and $m = -1$, $\nu = 0.25$ and the plate does not buckle in $0.4 \leqslant (r/a) \leqslant 1$.

5. Numerical Discussion

To have an idea of the effect of non-uniformity in thickness, we assume

$$\nu = 0.25,\ m = -1 \ \text{and}\ 16Eh_0^2 = 45Ta^2$$

then plot graphically the values of

$$M = \frac{\pi Th_0}{P}\,w \qquad \text{(for case I)}$$

and

$$N = \frac{2Th_0}{qa^2}\,w \qquad \text{(for case II)}$$

against R ($= r/a$) from 0.4 to 1. $\bar{M}$ and $\bar{N}$ in Figs. 1–3 give the values of M and N in the associated case of uniform thickness.

Figure 1 shows the deflection under boundary conditions (a) for case I, by M and $\bar{M}$ and for case II, by N and $\bar{N}$. Figure 2 shows the deflection under boundary conditions (b) for case I by M and $\bar{M}$. Figure 3 shows the deflection under boundary conditions (b) for case II by N and $\bar{N}$.

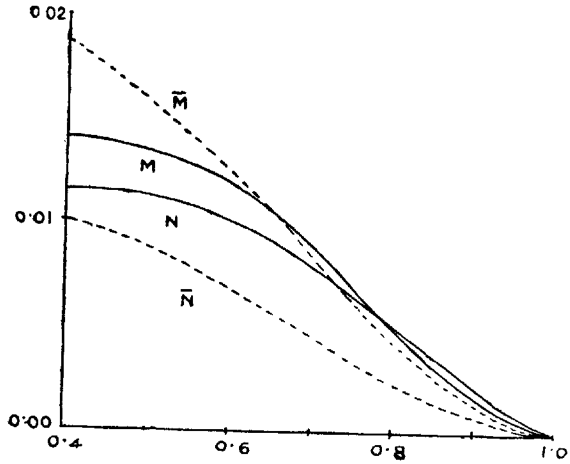

Fig. 1.

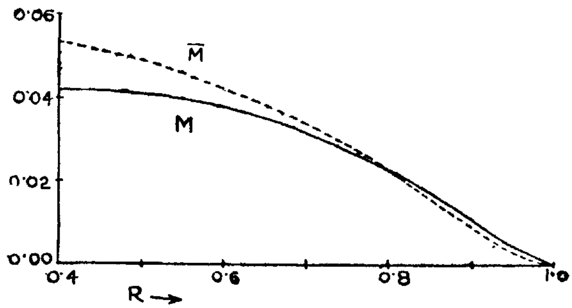

Fig. 2.

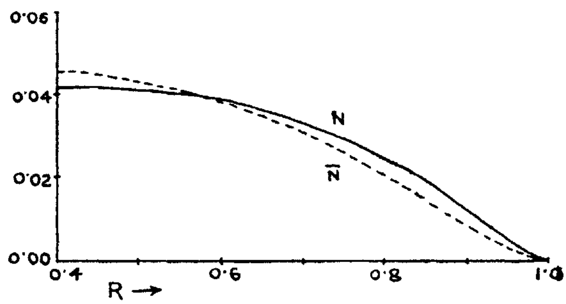

Fig. 3.

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REFERENCES

Conway, H. D. (1948). The bending of symmetrically loaded circular plate of variable thickness. *J. appl. Mech.*, **15**, 1.

———— (1951). Axially symmetric plates with linearly varying thickness. *J. appl. Mech.*, **18**, 140.

Dhaliwal, R. S. (1968). The bending of symmetrically loaded circular plates of variable thickness under uniform compression. *Aeronaut. Quart.*, **19**, 59.

Erdelyi, A. (1953). Higher Transcendental Functions, 2nd ed. McGraw-Hill Book Co., Inc., New York.

Fung, Y. C. (1953). Bending of thin elastic plate of variable thickness. *J. Aero. Sci.*, **20**, 455.

Holzer, H. (1918). *Zeitschrift für Turbinenwesen*, **15**, 21.

Mansfield, E. H. (1962). On the analysis of elastic plates of variable thickness. *Q. Jl Mech. appl. Math.*, **15**.

Olsson, R. G. (1937). Biegung dreisformiger Platten von radial verander licher Steifigkeit. *Ing. Archiv*, **8**, 81.

Pichler, O. (1928). Die Biegung Krieissymmetrischer Platten von veranderlicher Dicke (Dissertation). Stuttgart, Berlin.

Timoshenko, S., and Woinosky-Krieger, S. (1959). Theory of Plates and Shells, 2nd ed. McGraw-Hill Book Co., Inc., New York.