Indian J. pure appl. Math., 23(2): 129-130, February 1992

REMARK ON PACHPATTE'S GENERALIZATION OF AN HARDY'S INEQUALITY

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(*Received* 2 *May* 1991; *accepted* 11 *October* 1991)

In this note we show that Pachpatte's generalization of Hardy's inequality from Pachpatte[1] is a simple consequence of two inequalities: Jensen's inequality for convex functions and Copson's generalization of Hardy's inequality.

Pachpatte[1] proved the following result:

Let $H(u)$ be a real-valued non-negative convex function defined for $u \geqq 0$. Let $f(x) \geqq 0$, $r(x) > 0$ be real-valued and integrable functions defined on $(0, \infty)$ and

$$R(x) = \int_0^x r(t)\, dt,\ F(x) = \int_0^x r(t)\, f(t)\, dt$$

exist for $x > 0$. If $p \geqq 1$, $q > 1$ and

$$\int_0^{+\infty} r(x) R^{p-q}(x)\, H^p(f(x))\, dx < \infty$$

then

$$\int_0^{+\infty} r(x) R^{p-q}(x) H^p\left(\frac{F(x)}{R(x)}\right) dx \leqq \left(\frac{p}{q-1}\right)^p \int_0^{+\infty} r(x) R^{p-q}(x) H^p(f(x))\, dx. \quad \ldots(1)$$

Pachpatte noted that in the special case, when $H(u) = u$, inequality (1) reduces to the following inequality

$$\int_0^{+\infty} r(x) R^{-q}(x) F^{p'}(x) dx \leqq \left(\frac{p}{q-1}\right)^p \int_0^{+\infty} r(x) R^{p-q}(x) f^p(x) dx \quad \ldots(2)$$

which is established by Copson[2] (Th. 1).

But the reverse implication is also valid, i.e. we can give a very simple proof of (1) by using (2) and the well-known Jensen inequality for convex functions.

Indeed, since H is convex, by Jensen's inequality we have

$$\int_0^{+\infty} r(x)R^{p-q}(x)H^p\left(\frac{F(x)}{R(x)}\right)dx \geqq \int_0^{+\infty} r(x)R^{p-q}(x)\left(\frac{Q(x)}{R(x)}\right)^p dx$$

$$= \int_0^{+\infty} R^{-q}(x)r(x)\ Q^p\ (x)\ dx \qquad \ldots(3)$$

where

$$Q(x) = \int_0^x r(t)H(f(t))\ dt.$$

Further, by substitution $f(x) \to H(f(x))$ we get from (2) :

$$\int_0^{+\infty} r(x)R^{-q}\ Q^p(x)\ dx \leqq \left(\frac{p}{q-1}\right)^p \int_0^{+\infty} r(x)R^{p-q}(x)H^p(f(x))\ dx. \quad \ldots(4)$$

Now, from (3) and (4) we directly obtain (1).

Reerences

1. B. G. Pachpatte, *Indian J. pure appl. Math.* **21** (1990), 617-20.
2. E. T. Copson, *Proc. R. Soc. Edinburgh* **75S 13** (1975-76), 157-64.