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ON GRONWALL LIKE INTEGRAL INEQUALITIES

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In this paper we establish some new Gronwall like integral inequalities in one and two independent variables which can be used in the analysis of various problems in the theory of certain classes of ordinary and partial differential and integral equations.

1. INTRODUCTION

In his search for the existence of solutions of wave equations with logarithmic nonlinearities Haraux[6] was led to use the Gronwall like inequality derived in Corollary 16 (p. 139). In the study of global regular solutions for the dynamic antiplane shear problem in nonlinear viscoelasticity. Engler[4] used the following slight variant of the inequality given by Haraux[6] (Corollary 16, p. 139).

Lemma – Let $c > 0$, $a \in L^1(0, T, R^+)$, and assume that the function $w : [0, T] \to [1, \infty)$ satisfies

$$w(t) \leq c\left(1 + \int_0^t a(s)\, w(s) \log w(s)\, ds\right), \quad 0 \leq t \leq T \qquad \ldots(1)$$

then

$$w(t) \leq c \exp\left(c \int_0^t a(s)\, ds\right), \quad 0 \leq t \leq T. \qquad \ldots(2)$$

An essential difference between the above inequality and the well known Gronwall's inequality[7] (see references 1, 2, 5, 10) is that it contains an extra logarithmic factor in the integrand on the right side in (1). In view of the applications of the above inequality given in (see references 4, 6), and other applications, one is naturally led to look for generalizations and extensions of the inequality given in Lemma which are applicable to some new classes of differential and integral equations. Our main aim in this paper is to give some new generalizations and extensions of the inequality given in Lemma which can be used as handy tools in the study of certain classes of differential and integral equations in one and two independent variables.

2. STATEMENT OF RESULTS

We first give some basic notations and definitions which will be used in our subsequent discussion. Let $I = [0, T]$, $T > 0$ is finite but can be arbitrarily large.

Let R denote the set of real numbers and $R_+ = [0, \infty)$, $R_+^0 = (0, \infty)$ and $R_1 = [1, \infty)$. We write $p \in L^1(I, R)$ whenever p is measurable from $I \to R$ and

$$\int_0^T |p(t)|\, dt < \infty.$$

We define the differential operators L_i, $0 \le i \le n$ by

$$L_0 x(t) = x(t),\ L_i x(t) = \frac{1}{p_i(t)} \frac{d}{dt} (L_{i-1} x(t)),\quad 1 \le i \le n,$$

with $p_n(t) = 1$, where $x(t)$ and $p_i(t) > 0$ are some functions defined for $t \in I$. For $t \in I$ and some functions $q_j(t) > 0$, $j = 1, \ldots, n-1$ and $q(t) \ge 0$ defined for $t \in I$, we set

$$H[t, q_1, q_2, \ldots, q_{n-1}, q] = \int_0^t q_1(t_1) \int_0^{t_1} q_2(t_2) \cdots \int_0^{t_{n-2}} q_{n-1}(t_{n-1}) \int_0^{t_{n-1}} q(t_n)\, dt_n\, dt_{n-1} \ldots dt_2\, dt_1,$$

where $t_0 = t$.

Let $I_a = [0, a]$, $I_b = [0, b]$, $\Delta = I_a \times I_b$ for $a, b \in R$, $a > 0$, $b > 0$, and $\Delta_1 = \{(x, y) : 0 \le x \le x_1, 0 \le y \le y_1\} \subset \Delta$. Let $m, n \ge 1$ be integers. We define the differential operators D_1^n and D_2^m by

$$D_1^n r(x, y) = \frac{\partial^n r(x, y)}{\partial x^n},\ D_2^m r(x, y) = \frac{\partial^m r(x, y)}{\partial y^m}$$

where $r(x, y)$ is some function defined for $(x, y) \in \Delta$. For $(x, y) \in \Delta$ and some function $k(x, y)$ defined for $(x, y) \in \Delta$, we set

$$M[x, y, k] = \int_0^x \int_0^{s_{n-1}} \cdots \int_0^{s_1} \int_0^y \int_0^{t_{m-1}} \cdots \int_0^{t_1} k(s, t)\, dt\, dt_1 \ldots dt_{m-1}\, ds\, ds_1 \ldots ds_{n-1}$$

where $s_0 = x$ and $t_0 = y$. It is easy to observe that

$$M[x, y, k] = M[y, x, k].$$

A fairly general version of the inequality given in Lemma is embodied in the following theorem.

Theorem 1 – Let $p_i \in L^1(I, R_+^0)$, $p \in L^1(I, R_+)$. Let g be a continuously differentiable function defined on R_+ and $g > 0$ on R_+^0 and $g(0) = 0$, $g' \ge 0$ on R_+^0. If $u : I \to R_1$ satisfies

$$u(t) \le u_0 + H[t, p_1, p_2, \ldots, p_{n-1}, pug(\log u)] \qquad \ldots(3)$$

for $t \in I$, where $u_0 \ge 1$ is a constant, then for $0 \le t \le t_1$, $t, t_1 \in I$,

$$u(t) \le \exp G^{-1}[G(\log u_0) + H[t, p_1, p_2, \ldots, p_{n-1}, p]] \qquad \ldots(4)$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \; r_0 > 0, \; r \geq 0 \qquad \ldots(5)$$

G^{-1} is the inverse of G and t_1 is chosen so that

$$G(\log u_0) + H[t, p_1, p_2, \ldots, p_{n-1}, p] \in \text{Dom}\,(G^{-1})$$

for all t lying in the subinterval $0 \leq t \leq t_1$ of I.

An immediate consequence of Theorem 1 is the following

Corollary 1 – Let p_i, p, u_0 be as defined in Theorem 1. If $u : I \to R_1$ satisfies

$$u(t) \leq u_0 + H[t, p_1, p_2, \ldots, p_{n-1}, pu \log u] \qquad \ldots(6)$$

for $t \in I$, then

$$u(t) \leq u_0 \exp\{H[t, p_1, p_2, \ldots, p_{n-1}, p] \qquad \ldots(7)$$

for $t \in I$.

Another Corollary of Theorem 1 is :

Corollary 2 – Let p_i, p, u_0 be as defined in Theorem 1. If $u : I \to R_1$ satisfies

$$u(t) \leq u_0 + H[t, p_1, p_2, \ldots, p_{n-1}, pu \{\log u\}^{\alpha}] \qquad \ldots(8)$$

for $t \in I$, where $0 < \alpha < 1$ is a constant, then

$$u(t) \leq \exp\{[\{\log u_0\}^{1-\alpha} + (1-\alpha) H[t, p_1, p_2, \ldots, p_{n-1}, p]]^{1/1-\alpha}\} \qquad \ldots(9)$$

for $t \in I$.

Remark 1 : By setting $n = 1$ and $p(t) = u_0 q(t)$ in Theorem 1 we arrive at the inequality given in Lemma. For other useful variants of Lemma, see Haraux[6] (p. 139).

We next establish the following integral inequality in two independent variables which can be used in some applications.

Theorem 2 – Let $h \in L^1(\Delta, R_+)$ and g be as defined in Theorem 1. If $z : \Delta \to R_1$ satisfies

$$z(x, y) \leq z_0 + M[x, y, hzg(\log z)] \qquad \ldots(10)$$

for $(x, y) \in \Delta$, where $z_0 \geq 1$ is a constant, then for $(x, y) \in \Delta_1 \subset \Delta$

$$z(x, y) \leq \exp\{G^{-1} G(\log z_0) + M[x, y, h]]\} \qquad \ldots(11)$$

where G and G^{-1} are as defined in Theorem 1 and $(x, y) \in \Delta_1 \subset \Delta$ is chosen so that

$$G(\log z_0) + M[x, y, h] \in \text{Dom}\,(G^{-1})$$

for all $(x, y) \in \Delta_1 \subset \Delta$.

The following corollaries can be obtained as immediate consequences of Theorem 2.

Corollary 3 – Let h and z_0 be as defined in Theorem 2. If $z : \Delta \to R_1$ satisfies

$$z(x, y) \leq z_0 + M[x, y, hz \log z] \quad \ldots(12)$$

for $(x, y) \in \Delta$, then

$$z(x, y) \leq z_0^{\exp\{M[x, y, h]\}} \quad \ldots(13)$$

for $(x, y) \in \Delta$.

Corollary 4 – Let h and z_0 be as defined in Theorem 2. If $z : \Delta \to R_1$ satisfies

$$z(x, y) \leq z_0 + M[x, y, hz\{\log z\}^{\alpha}] \quad \ldots(14)$$

for $(x, y) \in \Delta$, where $0 < \alpha < 1$ is a constant, then

$$z(x, y) \leq \exp\left\{\left[\{\log z_0\}^{1-\alpha} + (1-\alpha) M[x, y, h]\right]^{1/(1-\alpha)}\right\} \quad \ldots(15)$$

for $(x, y) \in \Delta$.

Remark 2 : In the special case when $n = 1$, $m = 1$, the inequality in Corollary 3 takes the following form.
If

$$z(x, y) \leq z_0 + \int_0^x \int_0^y h(s, t)\, z(s, t) \log z(s, t)\, dt\, ds \quad \ldots(16)$$

for $(x, y) \in \Delta$, then

$$z(x, y) \leq z_0^{\exp\left\{\int_0^x \int_0^y h(s, t)\, dt\, ds\right\}} \quad \ldots(17)$$

for $(x, y) \in \Delta$, where h, z, z_0 are as defined in Corollary 3. We note that this inequality is an extension of the inequality given in Lemma to two independent variables.

3. Proofs of Theorem 1 and Corollaries 1 and 2

Let $\epsilon > 0$ be arbitrary small constant and define $u_\epsilon(t) = u(t) + \epsilon$ for $t \in I$. We note that since g is differentiable on R_+^0 and $g' \geq 0$ on R_+^0, the function g is monotonically increasing on R_+^0. From (3) we see that

$$\begin{aligned} u_\epsilon(t) &\leq u_0 + \epsilon + H[t, p_1, p_2, \ldots, p_{n-1}, p(u_\epsilon - \epsilon)\, g(\log(u_\epsilon - \epsilon))] \\ &\leq u_0 + \epsilon + H[t, p_1, p_2, \ldots, p_{n-1}, pu_\epsilon\, g(\log u_\epsilon)]. \end{aligned} \quad \ldots(18)$$

Define a function $v(t)$ by

$$v(t) = u_0 + \epsilon + H[t, p_1, p_2, \ldots, p_{n-1}, pu_\epsilon\, g(\log u_\epsilon)],\ v(0) = u_0 + \epsilon. \quad \ldots(19)$$

From (19) it is easy to observe that

$$L_n v(t) = p(t)\, u_\epsilon(t)\, g(\log u_\epsilon(t))]. \quad \ldots(20)$$

Using the fact that $u_\epsilon(t) \leq v(t)$ in (20) we have

$$L_n v(t) \leq p(t)\, v(t)\, g(\log v(t)). \qquad \ldots(21)$$

From (21) and using the facts that $v(t)$ is positive, $\frac{d}{dt} v(t)$ and $L_{n-1} v(t)$ are non-negative for $t \in I$, we observe that

$$\frac{L_n v(t)}{v(t)} \leq p(t)\, g(\log v(t)) + \frac{\left(\frac{d}{dt} v(t)\right) L_{n-1} v(t)}{v^2(t)}$$

i.e.

$$\frac{d}{dt}\left[\frac{L_{n-1} v(t)}{v(t)}\right] \leq p(t)\, g(\log v(t)). \qquad \ldots(22)$$

Integrating (22) from 0 to t and using the fact that $L_{n-1} v(0) = 0$, we obtain

$$\frac{L_{n-1} v(t)}{v(t)} \leq \int_0^t p(t_n)\, g(\log v(t_n))\, dt_n$$

which implies

$$\frac{\frac{d}{dt} L_{n-2} v(t)}{v(t)} \leq p_{n-1}(t) \int_0^t p(t_n)\, g(\log v(t_n))\, dt_n. \qquad \ldots(23)$$

Again as above from (23) we observe that

$$\frac{d}{dt}\left[\frac{L_{n-2} v(t)}{v(t)}\right] \leq p_{n-1}(t) \int_0^t p(t_n)\, g(\log v(t_n))\, dt_n$$

which upon integrating from 0 to t and using the fact that $L_{n-2} v(0) = 0$, we get

$$\frac{L_{n-2} v(t)}{v(t)} \leq \int_0^t p_{n-1}(t_{n-1}) \int_0^{t_{n-1}} p(t_n)\, g(\log v(t_n))\, dt_n\, dt_{n-1}.$$

Continuing in this way we obtain

$$\frac{\frac{d}{dt} v(t)}{v(t)} \leq p_1(t) \int_0^t p_2(t_2) \int_0^{t_2} p_3(t_3) \ldots \int_0^{t_{n-2}} p_{n-1}(t_{n-1}) \times \int_0^{t_{n-1}} p(t_n)\, g(\log v(t_n))\, dt_n\, dt_{n-1} \ldots dt_3\, dt_2. \qquad \ldots(24)$$

Integrating (24) from 0 to t we obtain

$$\log v(t) \leq \log (u_0 + \epsilon) + H [t, p_1, p_2, \ldots, p_{n-1}, pg (\log v)]. \quad \ldots(25)$$

Define a function $m(t)$ by

$$m(t) = \log (u_0 + \epsilon) + H [t, p_1, p_2, \ldots, p_{n-1}, pg (\log v)],$$

$$m(0) = \log (u_0 + \epsilon). \quad \ldots(26)$$

From (26) it is easy to observe that

$$L_n m(t) = p(t) g(\log v(t)). \quad \ldots(27)$$

Using the fact that $\log v(t) \leq m(t)$ in (27) we have

$$L_n m(t) \leq p(t) g(m(t)). \quad \ldots(28)$$

From (28) and using the facts that $g(m(t))$ is positive, $\frac{d}{dt} g(m(t)) \geq 0$, $L_{n-1} m(t) \geq 0$ for $t \in I$, we observe that

$$\frac{L_n m(t)}{g(m(t))} \leq p(t) + \frac{\left[\frac{d}{dt} g(m(t))\right] L_{n-1} m(t)}{g^2 (m(t))}$$

e.g.

$$\frac{d}{dt}\left[\frac{L_{n-1} m(t)}{g(m(t))}\right] \leq p(t). \quad \ldots(29)$$

Now repeating the above arguments below the inequality (22) up to the inequality (24) with suitable changes, we obtain

$$\frac{\frac{d}{dt} m(t)}{g(m(t))} \leq p_1 (t) \int_0^t p_2 (t_2) \int_0^{t_2} p_3 (t_3) \ldots \times \int_0^{t_{n-2}} p_{n-1} (t_{n-1}) \int_0^{t_{n-1}} p(t_n)\, dt_n\, dt_{n-1} \ldots dt_3\, dt_2. \quad \ldots(30)$$

From (5) and (30) we have

$$\frac{d}{dt} G(m(t)) \leq p_1 (t) \int_0^t p_2 (t_2) \int_0^{t_2} p_3 (t_3) \ldots \int_0^{t_{n-2}} p_{n-1} (t_{n-1}) \times \int_0^{t_{n-1}} p(t_n)\, dt_n\, dt_{n-1} \ldots dt_3\, dt_2. \quad \ldots(31)$$

Now keeping $t = t_1$ and integrating both sides of (31) from 0 to t we get

$$G(m(t)) \leq G(\log (u_0 + \epsilon)) + H[t, p_1, p_2, \ldots, p_{n-1}, p]. \quad \ldots(32)$$

Substituting the bound for $m(t)$ from (32) in (25) we have

$$\log v(t) \leq G^{-1}[G(\log (u_0 + \epsilon)) + H[t, p_1, p_2, \ldots, p_{n-1}, p]]$$

i.e.

$$v(t) \leq \exp\{G^{-1}[G(\log (u_0 + \epsilon)) + H[t, p_1, p_2, \ldots, p_{n-1}, p]]\}. \quad \ldots(33)$$

The desired bound in (4) now follows by substituting (33) in (18) and letting $\epsilon \to 0$. The subinterval for t is obvious. This completes the proof of Theorem 1.

If we take in Theorem 1, $g(s) = 2$, then (5) reduces to $G(r) = \log r/r_0$, which implies $r = r_0 \exp(G(r))$ and hence $G^{-1}(r) = r_0 \exp(r)$.

Using these facts it is easy to observe that

$$\begin{aligned} & G^{-1}[G(\log u_0) + H[t, p_1, p_2, \ldots, p_{n-1}, p]] \\ & = (\log u_0) \exp\{H[t, p_1, p_2, \ldots, p_{n-1}, p]\} \\ & = \log u_0^{\exp\{H[t, p_1, p_2, p_{n-1}, p]\}}. \end{aligned} \quad \ldots(34)$$

The desired inequality (7) in Corollary 1 follows from (4) and (34).

By taking $g(s) = s^\alpha$, $0 < \alpha < 1$ in Theorem 1 and following the proof of Corollary 1 given above with suitable modifications we get the required inequality (9) in Corollary 2.

4. Proofs of Theorem 2 and Corollaries 3 and 4

Let $\epsilon > 0$ be arbitrary small constant and define $z_\epsilon(x, y) = z(x, y) + \epsilon$ for $(x, y) \in \Delta$. Since g is differentiable on R_+^0 and $g' \geq 0$ on R_+^0, we observe that the function g is monotonically increasing on R_+^0. From (10) we see that

$$\begin{aligned} z_\epsilon(x, y) & \leq z_0 + \epsilon + M[x, y, h(z_\epsilon - \epsilon) g(\log (z_\epsilon - \epsilon))] \\ & \leq z_0 + \epsilon + M[x, y, hz_\epsilon g(\log z_\epsilon)]. \end{aligned} \quad \ldots(35)$$

Define a function $v(x, y)$ by

$$\begin{aligned} & v(x, y) = z_0 + \epsilon + M[x, y, hz_\epsilon g(\log z_\epsilon)], \\ & v(0, y) = v(x, 0) = z_0 + \epsilon. \end{aligned} \quad \ldots(36)$$

From (36) it is easy to observe that

$$D_2^m D_1^n v(x, y) = h(x, y) z_\epsilon(x, y) g(\log z_\epsilon(x, y)). \quad \ldots(37)$$

Using the fact that $z_\epsilon(x, y) \leq v(x, y)$ in (37) we have

$$D_2^m D_1^n v(x, y) \leq h(x, y) v(x, y) g(\log v(x, y)). \quad \ldots(38)$$

From (38) and the facts that

$$v(x, y) > 0, D_2 v(x, y) \geq 0 \text{ and } D_2^{m-1} D_1^n v(x, y) \geq 0$$

for $(x, y) \in \Delta$, we observe that

$$\frac{D_2^m D_1^n v(x, y)}{v(x, y)} \leq h(x, y)\, g(\log v(x, y)) + \frac{[D_2 v(x, y)]\, D_2^{m-1} D_1^n v(x, y)]}{v^2(x, y)}$$

i.e.

$$D_2\left[\frac{D_2^{m-1} D_1^n v(x, y)}{v(x, y)}\right] \leq h(x, y)\, g(\log v(x, y)). \qquad \ldots(39)$$

By keeping x fixed in (39), we set $y = t$ and then integrating with respect to t from 0 to y and using the fact that $D_2^{m-1} D_1^n v(x, 0) = 0$, we have

$$\frac{D_2^{m-1} D_1^n v(x, y)}{v(x, y)} \leq \int_0^y h(x, t)\, g(\log v(x, t))\, dt. \qquad \ldots(40)$$

From (40) and the facts that $v(x, y) > 0$, $D_2 v(x, y) \geq 0$ and $D_2^{m-2} D_1^n v(x, y) \geq 0$ for $(x, y) \in \Delta$, we observe that

$$D_2\left[\frac{D_2^{m-2} D_1^n v(x, y)}{v(x, y)}\right] \leq \int_0^y h(x, t)\, g(\log v(x, t))\, dt. \qquad \ldots(41)$$

By keeping x fixed in (41), set $y = t_1$ and then integrating with respect to t_1 from 0 to y and using the fact that

$$D_2^{m-2} D_1^n v(x, 0) = 0$$

we have

$$\frac{D_2^{m-2} D_1^n v(x, y)}{v(x, y)} \leq \int_0^y \int_0^{t_1} h(x, t)\, g(\log v(x, t))\, dt\, dt_1.$$

Continuing in this way we obtain

$$\frac{D_1^n v(x, y)}{v(x, y)} \leq \int_0^y \int_0^{t_{m-1}} \cdots \int_0^{t_1} h(x, t)\, g(\log v(x, t))\, dt\, dt_1 \ldots dt_{m-1}. \qquad \ldots(42)$$

From (42) and the facts that $v(x, y) > 0$, $D_1 v(x, y) \geq 0$ and $D^{n-1} v(x, y) \geq 0$ for $(x, y) \in \Delta$, we observe that

$$D_1\left[\frac{D_1^{n-1} v(x, y)}{v(x, y)}\right] \leq \int_0^y \int_0^{t_{m-1}} \cdots \int_0^{t_1} h(x, t) g(\log v(x, t))\, dt\, dt_1 \ldots dt_{m-1}. \qquad \ldots(43)$$

Now keeping y fixed in (43), set $x = s$ and then integrating with respect to s from 0 to x and using the fact that

$$D_1^{n-1} v(0, y) = 0,$$

we have

$$\frac{D_1^{n-1} v(x, y)}{v(x, y)} \leq \int_0^x \int_0^y \int_0^{t_{m-1}} \cdots \int_0^{t_1} h(s, t)\, g(\log v(s, t))\, dt\, dt_1 \ldots dt_{m-1}\, ds.$$

Continuing in this way, we obtain

$$\log v(x, y) \leq \log (z_0 + \epsilon) + M [x, y, hg(\log v)]. \qquad \ldots(44)$$

Define a function $f(x, y)$ by

$$f(x, y) = \log (z_0 + \epsilon) + M [x, y, hg (\log v)],$$

$$f(0, y) = f(x, 0) = \log (x_0 + \epsilon). \qquad \ldots(45)$$

From (45) it is easy to observe that

$$D_2^m D_1^n f(x, y) = h(x, y)\, g(\log v(x, y)). \qquad \ldots(46)$$

Using the fact that $\log v(x, y) \leq f(x, y)$ in (46) we have

$$D_2^m D_1^n f(x, y) \leq h(x, y)\, g(f (x, y)). \qquad \ldots(47)$$

Now by following the same steps as above with suitable modifications and closely looking at the proof of Theorem 1 given above and the proofs of Theorem 4 of Pachpatte[8] and Theorem 2 of Pachpatte[9] we get the desired inequality (11) in Theorem 2.

The details of the proofs of Corollaries 3 and 4 are very close to that of the proofs of Corollaries 1 and 2 given above with suitable changes and hence we omit it here.

In concluding, we note that the inequalities established in Corollaries 1 and 3 can be used to study the boundedness, uniqueness and other properties of the solutions of the following equations

$$\left.\begin{aligned} L_n x(t) &= p(t)\, x(t) \log |x(t)|, \\ L_{i-1} x(0) &= C_{i-1},\ i = 1, \ldots, n, \end{aligned}\right\} \qquad \ldots(48)$$

and

$$\left.\begin{aligned} D_2^m D_1^n u(x, y) &= h(x, y)\, u (x, y) \log |u(x, y)|, \\ \frac{\partial^j u(x, 0)}{\partial y^j} &= \alpha_j (x), \quad 0 \leq j \leq m - 1, \\ \frac{\partial^i u(0, y)}{\partial x^i} &= \beta_i (y), \quad 0 \leq i \leq n - 1 \end{aligned}\right\} \qquad \ldots(49)$$

respectively, under some suitable conditions on the functions involved in (48) and (49). For similar applications, see Pachpatte[8,9]. We also note that the inequality established in Theorem 2 can be extended very easily to more than two independent variables. The precise formulation of such a result and its proof is very close to that of the result given in Theorem 2 with suitable modifications, we do not discuss it here.

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