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THE GRAPHOIDAL COVERING NUMBER OF UNICYCLIC GRAPHS

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A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G such that every path in ψ has atleast two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is in exactly one path in ψ. The graphoidal covering number $\gamma(G)$ of G is defined to be the minimum cardinality of a graphoidal cover of G. In this paper we determine the graphoidal covering number of unicyclic graphs.

1. INTRODUCTION

By a graph we mean a finite undirected graph without loops or multiple edges. For graph theoretic terminology we refer to Harary[3].

If $P = (v_0, v_1, \ldots, v_n)$ is a path or a cycle in a grpah G, $v_1, v_2, \ldots, v_{n-1}$ are called internal vertices of P. The number of internal vertices of P is denoted by $t(P)$. The concept of graphoidal cover was introduced by Acharya and Sampathkumar[1].

Definition 1.1 – Let $G = (V, E)$ be a graph. A graphoidal cover of G is a set ψ of (not necessarily open) paths in G satisfying the following conditions:

(i) Every path in ψ has atleast two vertices.

(ii) Every vertex of G is an internal vertex of at most one path in ψ.

(iii) Every edge of G is in exactly one path in ψ.

Let $\mathcal{G}$ denote the set of all graphoidal covers of G. Then

$$\gamma(G) = \min_{\psi \in \mathcal{G}} |\psi|$$

is called the graphoidal covering number of G.

In this paper we determine the graphoidal covering number of unicyclic graphs. We need the following theorem.

Theorem 1.2[4] – Let T be a tree with n vertices of degree 1. Then $\gamma(T) = n-1$.

2. Main Results

For any graphoidal cover ψ of a graph G let t_ψ denote the number of vertices of G which are not internal vertices of any path in ψ. Let $t = \min t_\psi$, where the minimum is taken over all graphoidal covers of G.

Theorem 2.1 – $\gamma(G) = q - p + t$.

Proof : Let ψ be any graphoidal cover of G.

$$\text{Then } q = \sum_{p \in \psi} |E(P)|$$

$$= |\psi| + \sum_{p \in \psi} t(P)$$

$$= |\psi| + p - t_\psi.$$

Hence $|\psi| = q - p + t_\psi \geq q - p + t$ so that

$$\gamma(G) = q - p + t.$$

Corollary 2.2[4] – If there exists a graphoidal cover ψ of G such that every vertex of G is an internal vertex of a path in ψ,

then $\gamma(G) = q - p$.

Corollary 2.3[4] – Let G be a graph of order p and size q having n vertices of degree 1. Then $\gamma(G) \geq q - p + n$. Furthermore, equality holds if and only if there exists a graphoidal cover ψ of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ.

In the following theorem we determine the graphoidal covering number of unicyclic graphs. This result has been independently obtained by Devadas Acharya[2]; we give a different proof here.

Theorem 2.4 – Let G be a unicyclic graph with n vertices of degree 1. Let C be the unique cycle in G and let m be the number of vertices of degree greater than 2 on C.

Then

$$\gamma(G) = \begin{cases} 1 & \text{if } m = 0 \\ n + 1 & \text{if } m = 1 \text{ and } d(v) = 3 \text{ where } v \text{ is the unique vertex of degree greater than 2 on } C \\ n & \text{otherwise} \end{cases}$$

Proof : *Case i* : $m = 0$.

Then $G = C$ and hence $\gamma(G) = 1$.

Case ii : $m = 1$ and $d(v) = 3$ where v is the unique vertex of degree greater than 2 on C.

Let e be an edge on C incident at v. Then $G - e$ is a tree with $n + 1$ vertices of degree 1. Hence by Theorem 1.2, $\gamma(G - e) = n$. Let ψ_1 be a minimum graphoidal cover

of $G - e$. Then $\psi_1 \cup \{P\}$ where P is a path of length 1 consisting of the edge e is a graphoidal cover of G. Hence $\gamma(G) \leq n + 1$. Now, let ψ be any graphoidal cover of G.

Then all the n vertices of degree 1 and atleast one vertex on C are not internal vertices of any path in ψ.

Hence $t_\psi \geq n + 1$ so that $t = \min t_\psi \geq n + 1$.

Now by Theorem 2.1, $\gamma(G) = q - p + t \geq q - p + n + 1$.

Since G is unicyclic $q = p$ so that $\gamma(G) \geq n + 1$.

Hence $\gamma(G) = n + 1$.

Case iii : $m = 1$ and $d(v) > 3$ where v is the unique vertex of degree greater than 2 on C. Let G_1 be the subgraph of G obtained by deleting all the vertices other than v of the cycle C. G_1 is a tree with n vertices of degree 1 and hence by Theorem 1.2, $\gamma(G_1) = n - 1$. Let ψ_1 be a minimum graphoidal cover of G_1. Then $\psi_1 \cup \{C\}$, where v is taken as the initial and terminal vertex of the cycle C, is a graphoidal cover of G and every vertex of degree greater than one is an internal vertex of some path in ψ. Hence by Corollary 2.3, $\gamma(G) = n$.

Case iv : $m > 1$. Let v, w be two vertices of degree > 2 on C such that all vertices in a $v - w$ section of C other than v, w have degree 2. Let P denote this $v - w$ section. If P has length 1, let $G_1 = G - \{e\}$ where e is the edge vw. Otherwise let G_1 be the subgraph of G obtained by deleting all the internal vertices of P. Clearly G_1 is a tree with n vertices of degree 1 and hence $\gamma(G_1) = n - 1$. If ψ_1 is a minimum graphoidal cover of G_1, then $\psi_1 \cup \{P\}$ is a graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence by Corollary 2.3 $\gamma(G) = n$.

References

1. B. Devadas Acharya and E. Sampathkumar, *Indian J. pure appl. Math.* **18** (1987), 882-90.
2. B. Devadas Acharya, Further results on the graphoidal cover of graph, Preprint.
3. F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass., 1969.
4. C. Pakkiam and S. Arumugam, *Indian J. pure appl. Math.* **20** (1989), 330-33.