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ON THE EVALUATION OF ZETA FUNCTION $\zeta(2s)$

N. R. BHATTACHARJEE

Department of Mathematics, University of Chittagong, Bangladesh

AND

T. BHATTACHARJEE

Department of Mathematics, B.I.T., Chittagong, Bangladesh

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An interesting method to evaluate the value of $\zeta(m)$ for even m, m being finite, has been derived. In this connection we proved a theorem giving a direct expression regarding the evaluation of $\zeta(2s)$ for any non-negative integral s.

1. INTRODUCTION

We have proved the following theorem.

Theorem – For any integral, s

$$\zeta(2s) = \underset{n\to\infty}{\text{Lt}} \frac{\pi^{2s}}{n^{2s}} R_s$$

where R_s is the sum of the sth power of the roots of the equation

$$x^{(n-1)/2} - \frac{n^2-1}{6} x^{(n-3)/2} + \ldots + (-1)^{(n-1)/2} \frac{1}{n} 2^{n-1} = 0$$

where n is odd.

Remark : In a earlier article, the authors[3] present an expression which gives an algorithm for determining $\zeta(m)$ using which we can assess the value of $\zeta(m)$ for any even integral values of m.

The present result is simpler and direct one, from which the value of zeta function $\zeta(2s)$ can be evaluated. Also since $\zeta(2s)$ is related with B_{2s} the result is equally applicable for B_{2s}.

2. PROOF OF THEOREM

To prove the theorem we shall prove the following Lemma.

Lemma – $$\zeta(2s) = \underset{n\to\infty}{\text{Lt}} \frac{\pi^{2s}}{n^{2s}} \sum_{r=1}^{(n-1)/2} \text{cosec}^{2s} \frac{r\pi}{n}$$

where n is odd.

PROOF : From $\sin\phi < \phi < \tan\phi$ we can write

$$\frac{1}{\phi} < \operatorname{cosec}\phi \;\&\; \cot\phi < \frac{1}{\phi}$$

$$\therefore \frac{1}{\phi^{2s}} < \operatorname{cosec}^{2s}\phi < \left(1 + \frac{1}{\phi^2}\right)^s$$

$$\text{i.e. } \sum_{r=1}^{(n-1)/2} \frac{1}{r^{2s}} \frac{n^{2s}}{\pi^{2s}} < \sum_{r=1}^{(n-1)/2} \operatorname{cosec}^{2s}\frac{r\pi}{n} < \frac{n-1}{2} + \sum_{r=1}^{(n-1)/2} \frac{sn^2}{r^2\pi^2}$$

$$+ \sum_{r=1}^{(n-1)/2} \frac{s(s-1)}{2} \frac{n^4}{r^4\pi^4} + \ldots + \sum_{r=1}^{(n-1)/2} \frac{1}{r^{2s}} \frac{n^{2s}}{\pi^{2s}}.$$

$$\text{or } \sum_{r=1}^{(n-1)/2} \frac{1}{r^{2s}} < \frac{\pi^{2s}}{n^{2s}} \sum_{r=1}^{(n-1)/2} \operatorname{cosec}^{2s}\frac{r\pi}{n} < \frac{\pi^{2s}}{n^{2s}}\left(\frac{n-1}{2}\right)$$

$$+ \frac{\pi^{2s}}{n^{2s}} \frac{sn^2}{\pi^2} \sum_{r=1}^{(n-1)/2} \frac{1}{r^2} + \ldots + \sum_{r=1}^{(n-1)/2} \frac{1}{r^{2s}}$$

$$\text{or } \sum_{r=1}^{(n-1)/2} \frac{1}{r^{2s}} < \frac{\pi^{2s}}{n^{2s}} \sum_{r=1}^{(n-1)/2} \operatorname{cosec}^{2s}\frac{r\pi}{n} <$$

$$\frac{\pi^{2s}}{n^{2s}} \sum_{r=1}^{(n-1)/2} \sum_{l=0}^{s-1} \binom{s}{l} \left\{\frac{n}{r\pi}\right\}^{2l} + \sum_{r=1}^{(n-1)/2} \frac{1}{r^{2s}} \text{ ; where } \binom{s}{l} = {}^sC_l.$$

But $\dfrac{\pi^{2s}}{n^{2s}} \sum_{r=1}^{(n-1)/2} \sum_{l=0}^{s-1} \binom{s}{l} \left\{\frac{n}{r\pi}\right\}^{2l}$ tends to zero as $n \to \infty$

which implies that $\zeta(2s) = \underset{n\to\infty}{\text{Lt}} \dfrac{\pi^{2s}}{n^{2s}} \sum_{r=1}^{(n-1)/2} \operatorname{cosec}^{2s}\frac{r\pi}{n}$

which is the Lemma 1.

Lemma 2 – For odd n if $R_s = \sum_{r=1}^{(n-1)/2} \operatorname{cosec}^{2s}\frac{r\pi}{n}$ is the sum of the sth power of the roots of the equation

$$x^{(n-1)/2} - \frac{n^2-1}{6} x^{(n-3)/2} + \ldots (-1)^{(n-1)/2} \frac{1}{n} 2^{n-1} = 0.$$

Then $R_s = -(P_1 R_{s-1} + P_2 R_{s-2} + \ldots + sP_s)$

where $P_r = \dfrac{(-1)^r (n^2-1)(n^2-3^2) \ldots\ldots \{n^2-(2r-1)^2\}}{(2r+1)!}$.

PROOF : Since for odd n (Jha and Sinha[5])

$$\sin n\theta = n \sin\theta - \frac{n(n^2-1)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta + \ldots + (-1)^{(n-1)/2} 2^{n-1}\sin^n\theta.$$

$$= a_1 \sin\theta + a_3 \sin^3\theta + \ldots + a_{n-2}\sin^{n-2}\theta + a_n \sin^n\theta$$

$$= \Sigma\, a_r \sin^r\theta$$

where r includes all odd integers; $a_1 = n$; $a_n = 2^{n-1}$

and $a_{r+2} = -\dfrac{n^2-r^2}{(r+1)(r+2)}\, a_r$.

We see that $\pm \sin\dfrac{r\pi}{n}$, for $r = 1, 2, \ldots \dfrac{n-1}{2}$ are the roots of equation

$$1 - \frac{n^2-1}{6}\zeta^2 + \frac{(n^2-1^2)(n^2-3^2)}{5!}\zeta^4 + \ldots\ldots + (-1)^{(n-1)/2}\frac{1}{n}2^{n-1}\zeta^{n-1} = 0$$

so that $\operatorname{cosec}^2\dfrac{r\pi}{n}$ for $r = 1$ to $\dfrac{n-1}{2}$ are the roots of the equation

$$x^{(n-1)/2} - \frac{n^2-1}{6}x^{(n-3)/2} + \ldots. (-1)^{(n-1)/2}\frac{1}{n}2^{n-1} = 0$$

$$\therefore \sum_{r=1}^{(n-1)/2} \operatorname{cosec}^2\frac{r\pi}{n} = \text{sum of the roots} = \frac{n^2-1}{6}$$

similarly

$$\sum_{r=1}^{(n-1)/2} \operatorname{cosec}^4\frac{r\pi}{n} = \text{sum of the squares of the roots} = \frac{(n^2-1)(n^2+1)}{90}$$

$$\sum_{r=1}^{n-1} \operatorname{cosec}^6\frac{r\pi}{n} = \text{sum of the cubes of the roots}$$

$$= \frac{(n^2-1)(2n^4+23n^2+191)}{1890}$$

$$\sum_{r=1}^{(n-1)/2} \operatorname{cosec}^6\frac{r\pi}{n} = \text{sum of the cubes of the roots}$$

$$= \frac{(n^2-1)(2n^4+23n^2+191)}{1890}$$

... ...

... ...

and so on.

Thus if we define R_s as

$$R_s = \sum_{r=1}^{(n-1)/2} \operatorname{cosec}^{2s}\frac{r\pi}{n}$$ and remembering the relation [see Appendix]

$$R_s = P_1 R_{s-1} + P_2 R_{s-2} + \ldots. + sP_s = 0$$

where

$$P_1 = -\frac{(n^2-1)}{6},\ P_2 = \frac{(n^2-1)(n^2-3^2)}{5!},$$

$$P_3 = -\frac{(n^2-1)\ (n^2-3^2)\ (n^2-5^2)}{7!}$$

... ...

... ...

$$P_r = \frac{(-1)^r\ (n^2-1)\ (n^2-3^2)\ \ldots\ \{n^2-(2r-1)^2\}}{(2r+1)!}$$

... ...

... ...

we get Lemma 2.

2.1. From Lemma 1 and 2 the theorem follows.

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APPENDIX

The equation

$$x^{(n-1)/2} - \frac{n^2-1}{6}\ x^{(n-3)/2} + \ldots.\ (-1)^{(n-1)/2}\ \frac{1}{n}\ 2^{(n-1)} = 0 \qquad \ldots(i)$$

can be written as

$$x^{(n-1)/2} + P_1\ x^{(n-3)/2} + P_2\ x^{(n-5)/2} + P_3\ x^{(n-7)/2} + \ldots$$
$$P_r\ x^{\{n-(2r+1)\}/2} + \ldots P_n = 0 \qquad \ldots(ii)$$

or $$x^k + P_1\ x^{k-1} + P_2\ x^{k-2} + \ldots + P_n = 0 \qquad \ldots(iii)$$

But since this equation is derived from the expression

$$a_1 \sin\theta + a_3 \sin^3\theta + \ldots + a_{n-2} \sin^{n-2}\theta + a_n \sin^n\theta \qquad \ldots(\text{iv})$$

P_is stand as follows

$$\left.\begin{aligned} P_1 &= a_3 = -\frac{n^2-1}{6}, \quad P_2 = a_5 = -\frac{(n^2-1)(n^2-3^2)}{5!}, \\ P_3 &= -\frac{(n^2-1)(n^2-3^3)(n^2-5^2)}{7!} \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ P_r &= a_{2r+1} = \frac{(-1)^r(n^2-1)(n^2-3^2)\ldots\{n^2-(2r+1)^2\}}{(2r+1)!} \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}\right\} \qquad \ldots\ldots(\text{v})$$

Now from Newton's theorem for the sums of powers of the roots of an equation we get that (Bernard and Child[6]) if R_s is the sum of the sth powers of the roots of the equation (iii) then

$$R_s + P_1R_{s-1} + P_2R_{s-2} + \ldots + sP_s = 0. \qquad \ldots(\text{vi})$$

Also $\operatorname{cosec}^2 \frac{r\pi}{n}$ for $r = 1$ to $\frac{n-1}{2}$ (or 1 to k) are the $\frac{n-1}{2}$ (or k) numbers of roots of the equations (ii) (or(iii)).

Hence $R_s = \sum_{n=1}^{(n-1)/2} \operatorname{cosec}^{2s} \frac{r\pi}{n}$

satisfying the relation (vi), with P_is as in (v).