Indian J. pure appl. Math., 23(5): 325-332, May 1992

ON A TRANSFORMATION ASSOCIATED WITH SETS OF n-FUNDAMENTAL FORMS OF FINSLER HYPERSURFACES

U. P. Singh and Raj Kumar Srivastava

Department of Mathematics, University of Gorakhpur
Gorakhpur 273009

(*Received* 26 *July* 1991; *after revision* 9 *February* 1992; *accepted* 24 *February* 1992)

The purpose of the present paper is to find the relation between the v-curvature tensor with respect to the Cartan connection of Finsler spaces (M^n, L) and $(M^n, {}^*L)$ where ${}^*L(x, y)$ is obtained from $L(x, y)$ by a change ${}^*L = f(L, \beta)$, $\beta = b_i(x, y)\, y^i$ and $b_i(x, y)$ is an h-vector in (M^n, L). The relation in n-fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and $(M^n, {}^*L)$ has also been obtained.

1. Introduction

Let M^n be an n-dimensional differentiable manifold and $F^n = (M^n, L)$ be a Finsler space equipped with a fundamental function $L(x, y)$ $(y^i = \dot{x}^i)$ on M^n. Shibata[15] has studied the properties of a Finsler space $(M^n, {}'L)$ whose metric function ${}'L(x, y)$ is obtained from L by a change ${}'L(x, y) = f(L, \beta)$, where $\beta = b_i(x)\, y^i$ and f is a positively homogeneous function of degree one in L and β. This change of a metric function is called a β-change. We have specially interesting examples of β-changes of metrics, for instance

$$''L(x, y) = L(x, y) + \beta(x, y) \qquad \ldots(1.1)$$

$$'''L(x, y) = L^2(x, y)/\beta(x, y). \qquad \ldots(1.2)$$

The change (1.1) has been introduced by Matsumoto[8]. Hashiguchi and Ichijyō[4] named it a Randers change and proved a theorem which shows a relation between a Randers change and a projective change.

Next, the change (1.2) is called a Kropina change. For a β-change $L \to {}'L = f(L, \beta)$ if L is a Riemannian metric $\alpha(x, y) = (a_{ij}(x)\, y^i y^j)^{1/2}$, then ${}'L = f(L, \beta)$ becomes a well known (α,β)-metric[2,3].

In particular ${}'L = \alpha + \beta$ is a Randers metric [1,5] and ${}'L = \alpha^2/\beta$ is a Kropina metric[7,14]. Both of them are closely related to physics and so Finsler spaces with these metrics have been studied by many authors, from various stand-points in the physical and mathematical aspects[1,5,7,14,16,17].

The n-fundamental forms of a Riemannian hypersurface of a Riemannian space have been defined and their properties have been studied by Rund[12,13]. Prasad[11] has

obtained the relation in n-fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and $(M^n, {}^{''}L)$ whose metric functions are related by (1.1). In all these works the functions b_i are assumed to be a function of coordinates only.

Izumi[6] while studying the conformal transformation of Finsler spaces, introduced the h-vector $b_i(x,y)$ which is v-covariantly constant with respect to the Cartan connection $C\Gamma$ and satisfies $LC^h_{ij}\, b_h = \rho\, h_{ij}$ where $C^h_{ij} = g^{hk}\, C_{ijk}$ is Cartan's C-tensor, h_{ij} is the angular metric tensor, ρ is given by $\rho = LC^i b_i/(n-1)$ and C^i is the torsion vector $C^i_{jk}\, g^{jk}$. Thus the h-vector $b_i(x, y)$ is not only a function of coordinates but it is a function of directional arguments satisfying $L\partial b_i/\partial y^j = \rho\, h_{ij}$. It should be noted that ρ depends only on coordinates.

The terminology and notations are referred to well-known Mstsumoto's book[10].

2. The Finsler Space $(M^n, {}^*L)$

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space with a fundamental function $L(x, y)$. We consider a change of a Finsler metric which is defined by

$$L \to {}^*L = f(L, \beta) \qquad \ldots(2.1)$$

and have another Finsler space ${}^*F^n = (M^n, {}^*L)$ with ${}^*L = f(L, \beta)$ where $\beta = b_i(x, y)\, y^i$, $b_i(x, y)$ is an h-vector in (M^n, L) and $f(L, \beta)$ is a positively homogeneous function of L and β of degree one. In this case we can also proceed with calculation in the same way as in Shibata[15]. We put

$$f_1 = \partial f/\partial L,\ f_2 = \partial f/\partial \beta,\ f_{11} = \partial^2 f/\partial L\, \partial L \text{ etc.},$$

$$\partial_i = \partial/\partial x^i,\ \dot\partial_i = \partial/\partial y^i.$$

Since ${}^*L = f$ is a positively homogeneous function of L and β of degree one, we have

$$f = f_1 L + f_2\beta,\ Lf_{12} + \beta f_{22} = 0,\ Lf_{11} + \beta f_{12} = 0. \qquad \ldots(2.2)$$

For the latter use we put

$$p = ff_1/L,\ q = ff_2,\ q_0 = ff_{22}. \qquad \ldots(2.3)$$

Paying attention to $l_i = \dot\partial_i L$, from (2.2) we have

$${}^*l_i = f_1 l_i + f_2 b_i. \qquad \ldots(2.4)$$

Differentiating this by y^j, we have the angular metric tensor ${}^*h_{ij} = {}^*L\dot\partial_i\dot\partial_j\, {}^*L$ of ${}^*F^n$:

$${}^*h_{ij} = p' h_{ij} + q_0\, m_i\, m_j \qquad \ldots(2.5)$$

where $p' = f(f_1 + \rho f_2)/L$ and the covariant vector m_i is defined by

$$m_i = b_i - \beta\, y_i/L^2. \qquad \ldots(2.6)$$

It is noted that the covariant vector m_i is a non-vanishing one, and is orthogonal to the supporting element y^i.

Now from (2.2), (2.4), (2.5) and (2.6) the fundamental tensor ${}^*g_{ij} = \dot\partial_i\dot\partial_j\,({}^*L^2/2)$ of ${}^*F^n$ is given by

$$^*g_{ij} = p' g_{ij} + p_0 b_i b_j + p_{-1} (b_i y_j + b_j y_i) + p'_{-2} y_i y_j \quad \ldots(2.7)$$

where we put

$$p_0 = q_0 + f_2^2,\ q_{-1} = f f_{12}/L,\ p_{-1} = q_{-1} + p f_2/f,$$
$$q_{-2} = f(f_{11} - f_1/L)/L^2,\ p_{-2} = q_{-2} + p^2/f^2,$$
$$p'_{-2} = p_{-2} - \rho q/L^3. \quad \ldots(2.8)$$

The reciprocal tensor $^*g^{ij}$ of $^*g_{ij}$ can be written as

$$^*g^{ij} = (1/p')\, g^{ij} - s'_0\, b^i b^j - s'_{-1} (b^i y^j + b^j y^i) - s'_{-2}\, y^i y^j \quad \ldots(2.9)$$

where we put

$$b^i = g^{ij} b_j,\ b^2 = g^{ij} b_i b_j,\ v = b^2 - \beta^2/L^2,$$
$$s'_0 = f^2 q_0/L^2 \tau' p',\ s'_{-1} = (f^2/p'\tau' L^2)\ (p_{-1} + \rho f_2^2/L),$$
$$\tau' = (f^2/L^2)\ (p' + v q_0),\ s'_{-2} = p'_{-2}/pp' - (s'_{-1}/p)\ (v p_{-1} - \rho q \beta/L^3). \quad \ldots(2.10)$$

From the homogeneity it follows that

$$q_0\beta + q_{-1} L^2 = 0,\ q_{-1}\beta + q_{-2} L^2 = -p$$
$$p_0\beta + p_{-1} L^2 = q,\ q\beta + pL^2 = f^2$$
$$p_{-1}\beta + p_{-2} L^2 = 0. \quad \ldots(2.11)$$

As to the torsion tensor $^*C_{ijk} = \dot\partial_k\ (^*g_{ij}/2)$ of $^*F^n$, from (2.7) and (2.11) we get

$$^*C_{ijk} = p' C_{ijk} + p'_{-1}\ (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j)/2 + p_{02}\, m_i m_j m_k/2 \quad \ldots(2.12)$$

where we put $p'_{-1} = p_{-1} + (\rho/L)\, p_0$ and $p_{02} = \partial p_0/\partial\beta$. From the definition of m_i, it is evident that

(a) $m_i\, l^i = 0$
(b) $m_i b^i = m_i m^i = b^2 - \beta^2/L^2$, where $m^i = g^{ij} m_j$
(c) $h_{ij}\, m^i = h_{ij}\, b^i = m_j$
(d) $C^h_{ij}\, m_h = L^{-1}\, \rho\, h_{ij}$. $\quad \ldots(2.13)$

Contracting (2.12) by $^*g^{hk}$ and using (2.13) we have

$$^*C^h_{ij} = C^h_{ij} - V^h_{ij}$$

where we put

$$V^h_{ij} = Q^h\ (p' C_{imj}\, b^m + p'_{-1}\, m_i\, m_j) - (p'_{-1}/2\, p')\ (h^h_i\, m_j + h^h_j\, m_i) - (m^h/p' - v\, Q^h)\ (p_{02}\, m_i\, m_j + p'_{-1}\, h_{ij})/2$$
$$Q^h = s'_0\, b^h + s'_{-1}\, y^h,\ h^h_i = g^{hr}\, h_{ir},\ m^h = g^{hr}\, m_r. \quad \ldots(2.14)$$

We shall now find the v-curvature tensor $^*S^i_{hjk}$ of $(M^n, {}^*L)$ which with respect to the

Cartan connection is defined as

$$
\begin{aligned}
{}^{*}S^{i}_{hjk} &= A_{(jk)} \{ {}^{*}C^{m}_{hk} \, {}^{*}C^{i}_{mj} \} \\
&= S^{i}_{hjk} + A_{(jk)} \{ C^{i}_{mk} \, V^{m}_{hj} - C^{m}_{hk} \, V^{i}_{mj} - V^{i}_{mk} \, V^{m}_{hj} \}
\end{aligned}
$$

where S^{i}_{hjk} is the v-curvature tensor of (M^n, L) and $A_{(jk)} \{ \cdots \}$ denotes the interchange of the indices j, k and subtraction.

3. A Hypersurface of (M^n, L)

Let $(M^{n-1}, \bar{L})$ be a hypersurface of (M^n, L) given by the equation

$$x^i = x^i (u^\alpha)^{\dagger}. \qquad \ldots(3.1)$$

Let us suppose that the functions (3.1) are at least of class C^3 in u^α and the projection factors $B^i_\alpha = (\partial x^i / \partial u^\alpha)$ are such that their matrix has maximal rank $n-1$. The fundamental metric funtion $\bar{L}(u, v)$ of the hypersurface is given by

$$\bar{L}(u^\alpha, v^\alpha) = L(x^i (u^\alpha), B^i_\alpha v^\alpha)$$

where v^α is the element of support for the hypersurface for which

$$y^i = B^i_\alpha v^\alpha. \qquad \ldots(3.2')$$

Thus if l^α denotes the normalized vector along the element of support then

$$l^i = B^i_\alpha l^\alpha. \qquad \ldots(3.2)$$

If $g_{hj}(x, y)$ denotes the metric tensor of (M^n, L), the induced metric tensor of $(M^{n-1}, \bar{L})$ is given by

$$g_{\alpha\beta}(u, v) = g_{hj}(x, y) \, B^h_\alpha \, B^j_\beta. \qquad \ldots(3.3)$$

The inverse of (3.3) is denoted by $g^{\alpha\beta}(u, v)$ by means of which we define the quantities

$$B^\alpha_i (u, v) = g^{\alpha\beta}(u, v) \, g_{ij}(x, y) \, B^j_\beta. \qquad \ldots(3.4)$$

We choose a unit normal vector $N^j(x, y)$ of $(M^{n-1}, \bar{L})$ satisfying

$$g_{hj}(x, y) \, B^h_\beta N^j(x, y) = 0, \; g_{hj}(x, y) \, N^h(x, y) \, N^j(x, y) = 1. \qquad \ldots(3.5)$$

We have the following identities from (3.3), (3.4) and (3.5)

$$B^\alpha_j B^j_\beta = \delta^\alpha_\beta, \; B^j_\alpha B^\alpha_h + N^j N_h = \delta^j_h \qquad \ldots(3.6)$$

where $N_i = g_{ij}(x, y) \, N^j$.

If $C_{hjk}(x, y)$ denotes the $(h)hv$-torsion tensor of (M^n, L), the induced $(h)hv$-torsion tensor $C_{\alpha\beta\gamma}(u, v)$ of $(M^{n-1}, \bar{L})$ is given by

$$C_{\alpha\beta\gamma}(u, v) = C_{hjk}(x, y) \, B^h_\alpha \, B^j_\beta \, B^k_\gamma \qquad \ldots(3.7)$$

from which we obtain

$$C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk} \, B^j_\beta \, B^k_\gamma. \qquad \ldots(3.8)$$

---

†The Latin indices vary from 1 to n and Greek from 1 to $n-1$

The relative v-covariant derivative of the projection factor B^i_β with respect to the induced Cartan connection ICT is given in Matsumoto[9] by

$$B^i_\beta|_\gamma = - B^i_\alpha C^\alpha_{hk} + C^i_{hk} B^h_\beta B^k_\beta. \qquad \ldots(3.9)$$

This tensor is normal to $(M^{n-1}, \bar{L})$. Therefore we may write

$$B^i_\beta|_\gamma = M_{\beta\gamma} N^i. \qquad \ldots(3.10)$$

From (3.9) it is clear that $M_{\beta\gamma}$ is symmetric in β and γ and it may be written as

$$M_{\beta\gamma} = C_{ijk} N^i B^j_\beta B^k_\gamma. \qquad \ldots(3.11)$$

The tangent vector space M^{n-1}_x to M^{n-1} at every point x^i $(=u^\alpha)$ of the hypersurface is considered as the Riemannian space $(M^{n-1}_x, \bar{g}_x)$ with the Riemannian metric $\bar{g}_x = g_{\alpha\beta}(u, v)\, dv^\alpha\, dv^\beta$. The components of the $(h)hv$-torsion tensor $C^\alpha_{\beta\gamma}$ will be the Christoffel symbols associated with g_x. If M^n_x is the tangent vector space to M^n at x^i $(=u^\alpha)$, then $(M^{n-1}_x, \bar{g}_x)$ will be the hypersurface of (M^n_x, g_x) given by (3.2′), where $g_x = g_{ij}(x, y)\, dy^i\, dy^j$ is the Riemannian metric of M^n_x. The quantities $M_{\beta\gamma}$ given by (3.11) will be considered as the coefficients of the second fundamental form of the tangent Riemannian space $(M^{n-1}_x, \bar{g}_x)$.

In general the coefficients of the rth fundamental forms of $(M^{n-1}_x, \bar{g}_x)$ are defined as ([Rund[12]])

$$C_{(1)\alpha\beta} = g_{\alpha\beta}$$

$$C_{(2)\alpha\beta} = M_{\alpha\beta}$$

$$C_{(r)\alpha\beta} = C_{(r-1)\alpha\epsilon} M^\epsilon_\beta \quad (2 \le r \le n) \qquad \ldots(3.11')$$

where $M^\epsilon_\beta = g^{\alpha\epsilon} N_{\alpha\beta}$.

At the point of $(M^{n-1}, \bar{L})$, the vector field b_i may be written as

$$b_i = b_\alpha B^\alpha_i + \mu N_i \qquad \ldots(3.12)$$

where (a) $b_\alpha = b_i B^i_\alpha$, (b) $\mu = b_i N^i$. $\ldots(3.13)$

In the following we assume that b_i is tangential to $(M^{n-1}_x, \bar{L})$, so that

$$\text{(a)}\quad b_i = b_\alpha B^\alpha_i, \quad \text{(b)}\quad b^i = b^\alpha B^i_\alpha \qquad \ldots(3.14)$$

where $b^\alpha = g^{\alpha\beta} b_\beta$. $\ldots(3.15)$

If b_i is an h-vector in (M^n, L) then b_α is also so in $(M^{n-1}_x, \bar{L})$.

4. The n-Fundamental Forms of a Hypersurface of $(M^n, {}^*L)$

Let $(M^{n-1}, {}^*\bar{L})$ be a hypersurface of $(M^n, {}^*L)$ given by the same equation (3.1). The relations (2.7), (2.9), (3.2), (3.3) and (3.14) yield

$${}^*g_{\alpha\beta} = p'\, g_{\alpha\beta} + p_0 b_\alpha b_\beta + p_{-1} L(b_\alpha l_\beta + b_\beta l_\alpha)\, p'_{-2} L^2 l_\alpha l_\beta \qquad \ldots(4.1)$$

$${}^*g^{\alpha\beta} = (1/p')\, g^{\alpha\beta} - s'_0 b^\alpha b^\beta - s'_{-1} L(b^\alpha l^\beta + b^\beta l^\alpha) - s'_{-2} L^2 l^\alpha l^\beta \qquad \ldots(4.2)$$

where $l_\alpha = g_{\alpha\beta} l^\beta = l_i B^i_\alpha$.

From (2.6), (3.2), (3.3), (3.5) and (3.13a) we also have

$$h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \quad h_{ij} B^i_\alpha N^j = 0 \qquad \ldots(4.3)$$

$$m_\alpha = b_\alpha - \beta\, l_\alpha/\bar{L} = m_i B^i_\alpha. \qquad \ldots(4.4)$$

It is to be noted that even if ${}^*N^i$ is a unit normal vector to $(M^{n-1}, {}^*\bar{L})$ then it is not necessary normal to $(M^{n-1}, \bar{L})$. We may write

$${}^*N^i = N^\alpha B^i_\alpha + DN^i. \qquad \ldots(4.5)$$

To obtain N^α and D we use (2.7), (3.5) and (4.5). Thus we get

$${}^*g_{\alpha\beta} N^\alpha + p_0 D \mu b_\beta + D p_{-1} \mu L l_\beta = 0 \qquad \ldots(4.6)$$

$${}^*g_{\alpha\beta} N^\alpha N^\beta + p' D^2 + \mu (2p_0 D b_\beta N^\beta + 2 l_\alpha L N^\alpha D p_{-1} + p_0 D^2 \mu) = 1. \qquad \ldots(4.7)$$

If b_i is tangential to $(M^{n-1}, \bar{L})$ then $\mu = 0$ and the equations (4.6) and (4.7) give $N^\alpha = 0$, $D = (p')^{-1/2}$. From (4.5) it follows that

$${}^*N^i = (p')^{-1/2} N^i. \qquad \ldots(4.8)$$

Hence we have the following :

Theorem 4.1 – Let $(M^n, {}^*L)$ be the Finsler space obtained from (M^n, L) by a change (2.1). If $(M^{n-1}, {}^*\bar{L})$ and $(M^{n-1}, \bar{L})$ are hypersurfaces of these spaces and an h-vector b_i is tangential to the hypersurface $(M^{n-1}, \bar{L})$ then a vector normal to $(M^{n-1}, \bar{L})$ is also normal to $(M^{n-1}, {}^*\bar{L})$.

Now we establish the following:

Theorem 4.2 – Let $(M^n, {}^*L)$ be the locally Minkowskian space obtained from a locally Minkowskian space (M^n, L) by a change (2.1). Let $(M^{n-1}, {}^*\bar{L})$ and $(M^{n-1}, \bar{L})$ be hypersurfaces of $(M^n, {}^*L)$ and (M^n, L) respectively. If an h-vector b_i is tangential to the hypersurface $(M^{n-1}, \bar{L})$ and (M^n_x, g_x), $(M^n_x, {}^*g_x)$, $(M^{n-1}_x, \bar{g}_x)$, $(M^{n-1}_x, {}^*\bar{g}_x)$ are tangent Riemannian spaces to (M^n, L), $(M^n, {}^*L)$, $(M^{n-1}, \bar{L})$, $(M^{n-1}, {}^*\bar{L})$ respectively, then we have the following:

(i) The second fundamental forms of $(M^{n-1}_x, \bar{g}_x)$ and $(M^{n-1}_x, {}^*\bar{g}_x)$ are proportional.

(ii) Every asymptotic direction of $(M^{n-1}_x, \bar{g}_x)$ is an asymptotic direction of $(M^{n-1}_x, {}^*\bar{g}_x)$.

(iii) The rth fundamental tensors of $(M^{n-1}_x, \bar{g}_x)$ and $(M^{n-1}_x, {}^*\bar{g}_x)$ are related by

$${}^*C_{(r)\alpha\beta} = (p')^{(3-r)/2} \left[C_{(r)\alpha\beta} - \sum_{m=2}^{r-1} P_{(m)\beta} Q_{(r+1-m)\alpha} \right], \; 3 \le r \le n \qquad \ldots(4.9)$$

where $$P_{(m)\alpha} = \sqrt{(p' s_0')}\, C_{(m)\alpha\epsilon} b^\epsilon, \; 2 \le m \le n-1 \qquad \ldots(4.10)$$

$$R_{(m)} = \sqrt{(p' s_0')}\, P_{(m)\epsilon} b^\epsilon, \; 2 \le m \le n-1 \qquad \ldots(4.11)$$

$$Q_{(2)\alpha} = P_{(2)\alpha} \qquad \ldots(4.12a)$$

$$Q_{(r)\alpha} = P_{(r)\alpha} - \sum_{m=2}^{r-1} R_{(m)}\, Q_{(r+1-m)\alpha},\ 3 \leq r \leq n-1. \qquad \ldots(4.12b)$$

PROOF : (i) If b_i is tangential to the hypersurface $(M^{n-1}, \bar{L})$ then $\mu = 0$ and hence

$$m_i N^i = 0. \qquad \ldots(4.13)$$

Thus from (2.12), (3.11), (4.3), (4.8) and (4.13) it follows that

$$^*M_{\beta\gamma} = (p')^{1/2} M_{\beta\gamma}. \qquad \ldots(4.14)$$

This proves (i).

(ii) A direction t^α for which $M_{\alpha\beta}\, t^\alpha\, t^\beta = 0$ is said to be an asymptotic direction. In view of this definition and (4.14) we get (ii).

(iii) The validity of relation (4.9) is established by induction.

Since C_{ijk} is an indicatory tensor, from (3.2) and (3.11) it follows that $M_{\beta\gamma}\, l^\gamma = 0$. Hence from (3.11′) we get

$$C_{(r)\beta\gamma}\, l^\gamma = 0 = C_{(r)\beta\gamma}\, l^\beta,\ 2 \leq r \leq n. \qquad \ldots(4.15)$$

Hence from (4.10), (4.11) and (4.12) we get

$$P_{(r)\alpha}\, l^\alpha = 0,\ Q_{(r)\alpha}\, l^\alpha = 0,\ 2 \leq r \leq n. \qquad \ldots(4.16)$$

From (4.2), (4.14) and (4.15) we get

$$^*M^\alpha_\beta = (p')^{-1/2} [M^\alpha_\beta - p' (s'_0\, b^\alpha + s'_{-1}\, l^\alpha\, L)\, M_{\beta\gamma}\, b^\gamma]. \qquad \ldots(4.17)$$

The relation (3.11′), (4.10), (4.15), (4.16) and (4.17) yield

$$^*C_{(3)\alpha\beta} = C_{(3)\alpha\beta} - P_{(2)\alpha}\, P_{(2)\beta}. \qquad \ldots(4.18)$$

From (4.12a) and (4.18), it is evident that (4.9) holds for $r = 3$. For a given fixed value of the integer s with $3 \leq s \leq n-1$, we have

$$^*C_{(s+1)\alpha\beta} = {}^*C_{(s)\alpha\epsilon}\, {}^*M^\epsilon_\beta. \qquad \ldots(4.19)$$

Now let us suppose that (4.9) is valid for $r = 3, 4, 5, \ldots, s$ so that we can write (4.19) in the form

$$^*C_{(s+1)\alpha\beta} = (p')^{(3-s)/2} \left[C_{(s)\alpha\epsilon} - \sum_{m=2}^{s-1} P_{(m)\epsilon}\, Q_{(s+1-m)\alpha} \right] {}^*M^\epsilon_\beta$$

which in view of (4.10), (4.11), (4.12a), (4.15), (4.16) and (4.17) gives

$$^*C_{(s+1)\alpha\beta} = (p')^{(2-s)/2} \left[C_{(s+1)\alpha\beta} + \sum_{m=2}^{s-1} R_{(m)}\, Q_{(s+1-m)\alpha}\, P_{(2)\beta} - P_{(s)\alpha}\, P_{(2)\beta} - \sum_{m=2}^{s-1} P_{(m+1)\beta}\, Q_{(s+1-m)\alpha} \right]$$

$$= (p')^{(2-s)/2}\left[C_{(s+1)\alpha\beta} - \sum_{m=2}^{s-1} P_{(m+1)\beta}\, Q_{(s+1-m)\alpha} - P_{(2)\beta}\left\{P_{(s)\alpha} - \sum_{m=2}^{s-1} R_{(m)}\, Q_{(s+1-m)\alpha}\right\}\right]$$

$$= (p')^{(2-s)/2}\left[C_{(s+1)\alpha\beta} - \sum_{m=2}^{s} P_{(m)\beta}\, Q_{(s+2-m)\alpha}\right].$$

This shows that (4.9) is valid for $r = s + 1$, which completes the proof of (iii).

Acknowledgement

The second author is thankful to CSIR, New Delhi for the award of a research grant to carry out this research. He is also thankful to Dr B. N. Prasad for his valuable suggestions in the preparation of this paper.

References

1. H. A. Eliopoulos, *Acad. Roy. Belg. Bull. Cl. Sci.* (5) **51** (1965), 986-95.
2. M. Hashiguchi, S. Hōjō and M. Matsumoto, *J. Korean Math. Soc.* **10** (1973), 17-26.
3. M. Hashiguchi and Y. Ichijyō, *Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.)* **8** (1975), 39-46.
4. M. Hashiguchi and Y. Ichijyō, *Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.)*, **13** (1980), 33-40.
5. R. S. Ingarden, On the geometrically absolute optical representation in the electron microscope, *Trav. Soc. Sci. Lett. Wroslaw* **B 45** (1957), 60.
6. H. Izumi, *Tensor, M. S.* **34** (1980), 337-59.
7. M. Matsumoto, *Tensor, N. S.* **24** (1972), 29-37.
8. M. Matsumoto, *J. Math. Kyoto Univ.* **14** (1974), 477-98.
9. M. Matsumoto, *J. Math. Kyoto Univ.* **25** (1985), 107-44.
10. M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces,* Kaiseisha Press, Otsu, Japan, 1986.
11. B. N. Prasad, *Indian J. pure appl. Math.* **10** (1979), 1068-75.
12. H. Rund, *Tensor, N. S.* **22** (1971), 163-73.
13. H. Rund, *Anali di Mathemetica pura ed Applicata* **8** (1971), 99-122.
14. C. Shibata, *Rep. Math. Phys.* **13** (1978), 117-28.
15. C. Shibata, *J. Math. Kyoto Univ.* **24** (1984), 163-88.
16. C. Shibata, H. Shimada, M. Azuma and H. Yasuda, *Tensor, N. S.* **31** (1977), 219-26.
17. H. Yasuda and H. Shimada, *Rep. Math. Phys.* **11** (1977), 347-60.