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ON RELATIVE INTEGRAL BASES FOR PURE QUARTIC FIELDS

JOHN A. HYMO AND CHARLES J. PARRY

Department of Mathematics, Virginia Polytechnic Institute and State University Blacksburg, Virginia 24061, *U.S.A.*

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Necessary and sufficient conditions are given for a pure quartic field to have on integral basis over its quadratic subfields. Similar results are obtained for the normal closure of the pure quartic field. An integral basis is given for all these extensions whenever it exists.

1. INTRODUCTION

A relative extension of number fields may or may not have an integral basis. Let L denote the Galois closure of a pure quartic field and K, k be subfields of L such that $[K : k] = 2$. Explicit necessary and sufficient conditions are given for K/k to have an integral basis and a basis is determined whenever it exists.

General necessary and sufficient conditions for a relative quadratic extension of number fields to have an integral basis have been given by Mann[7]. Using Mann's criteria. Bird and Parry[2] gave explicit conditions for a bicyclic biquadratic field to have an integral basis over its quadratic subfields. Recently, the authors[4] gave explicit conditions for a cyclic quartic field to have an integral basis over its quadratic subfield. Similar results are obtained here for pure quartic fields.

2. NOTATION

$\Delta_{M/K}$: discriminant of M/K.

$\Delta_{M/K}(x_1, x_2, \ldots, x_n)$: relative discriminant of $x_1, x_2, \ldots, x_n$.

f, g, h : positive, squarefree, relatively prime integers such that $f > 1$ and h is odd.

$$\eta = \sqrt[4]{fg^2h^3},\ \bar{\eta} = \sqrt{fh},\ \bar{\bar{\eta}} = \sqrt[4]{f^3g^2h}$$

$$\alpha = \begin{cases} (1+i)\eta & g-\text{odd} \\ \left(\frac{1+i}{2}\right)\eta & g-\text{even} \end{cases}$$

$$\bar{\alpha} = i\bar{\eta}$$

$$\overline{\overline{\alpha}} = \begin{cases} (i-1)\,\overline{\overline{\eta}} & f,\ g-\text{odd} \\ \left(\dfrac{i-1}{2}\right)\overline{\overline{\eta}} & f \text{ or } g-\text{even} \end{cases}$$

$k_1 = Q(\sqrt{fh})$

$k_2 = Q(\sqrt{-fh})$

$k_3 = Q(\sqrt{-1})$

$K_1 = Q(\eta)$

$K_2 = Q((1+i)\eta)$

$K_3 = Q(\sqrt{fh}, \sqrt{-1})$

$L = Q(\eta, \sqrt{-1})$

B_i, b_i $(i = 1, 2)$: ideals of K_i and k_i respectively, whose square is 2, if they exist.

I : the ideal of k_i with $I^2 = (h)$.

$\mathfrak{p}_1 \cdot \mathfrak{p}_2 = (2)$: prime ideals of k_1 when $fh \equiv 1 \pmod 8$ such that $\mathfrak{p}_1$ ramifies in K_1 when g is odd.

$\hat{\mathfrak{p}}_1 \cdot \hat{\mathfrak{p}}_2 = (2)$: prime ideals of k_2 when $fh \equiv 7 \pmod 8$ such that $\hat{\mathfrak{p}}_1$ ramifies in K_2 when g is even.

$\mathfrak{p} = \mathfrak{p}_1$, $\hat{\mathfrak{p}} = \hat{\mathfrak{p}}_1$.

$P_1 \cdot P_2 = \mathfrak{p}_2$: prime ideals of K_1 which ramify in L when $fh \equiv 1 \pmod 8$ and g is odd.

$\hat{P}_1 \cdot \hat{P}_2 = \hat{\mathfrak{p}}_2$: prime ideals of K_2 which ramify in L when $fh \equiv 7 \pmod 8$ and g is even.

P, $\hat{P}$: prime ideals of K_1 and K_2 lying over 2.

$\epsilon_0 = a + b\sqrt{fh}$: fundamental unit of k_1.

$$\theta = \begin{cases} \sqrt[4]{fg^2} & h = 1. \\ \sqrt[4]{fg^2 h \epsilon_0^2} & h \neq 1 \text{ and } \pm h \text{ is a principal divisor of } k_1. \\ \sqrt[4]{-fg^2 h \epsilon_0^2} & h \neq 1 \text{ and } \pm 2h \text{ is a principal divisor of } k_1. \end{cases}$$

$$\gamma = \begin{cases} \dfrac{(g/2-\bar{\eta})(1-i)+2\sqrt[4]{-f(g/2)^2}}{4} & h=1 \text{ and } \pm 2 \text{ is not a principal divisor of } k_1. \\ \dfrac{(a\cdot g/2-\bar{\eta})(1-i)+2\sqrt[4]{-f(g/2)^2 h\epsilon_0^2}}{4} & h \neq 1 \text{ and } \pm h \text{ is a principal divisor of } k_1. \\ \dfrac{(b\cdot g/2-1)+(b\cdot g/2+1)\bar{\eta}+2\sqrt[4]{f(g/2)^2 h\epsilon_0^2}}{4} & \pm 2h \text{ is a principal divisor of } k_1. \end{cases}$$

$$\zeta = \begin{cases} \dfrac{(g+\bar{\alpha})\ (1-i)+2\theta}{4} & h = 1. \\ \dfrac{(ag+\bar{\alpha})\ (1-i)+2\theta}{4} & h \neq 1 \text{ and } \pm h \text{ is a principal divisor of } k_1. \end{cases}$$

$$\delta = \begin{cases} \dfrac{1+\bar{\eta}+\theta}{2} & \pm h \text{ is a principal divisor of } k_1. \\ \dfrac{1+i+\theta}{2} & \pm 2h \text{ is a principal divisor of } k_1. \end{cases}$$

$$\phi = \frac{h+\alpha \pm \bar{\alpha} \pm \bar{\bar{\alpha}}}{4} \quad \text{where the signs are chosen according as } g/2 \equiv \pm 1 \pmod 4.$$

$$\psi = \frac{1+\bar{\alpha}+(1+i)\epsilon_0\theta}{2}.$$

t : number of prime divisors of Δ_{k_1}/Q.

ρ : for a fixed real quadratic field k_1, ρ is the ratio of all pure quartic fields K_1 such that K_1/k_1 has an integral basis to all those fields K_1 which contain k_1 for a fixed value of g.

$\sim$: used to denote that two ideals of a field belong to the same ideal class.

3. Discriminants

In this section we compute Δ_{K_i/k_i} for $i = 1, 2$, $\Delta_{L/K}$, for $i = 1, 2, 3$ and Δ_{L/k_3}.

Theorem 1 – The relative discriminants are given by :

$$\Delta_{K_1/k_1} = \begin{cases} (2^2 g\sqrt{fh}) & fh \not\equiv 1 \pmod 8,\ g-\text{odd or } fh \equiv 1 \pmod 4,\ g-\text{even} \\ (g\sqrt{fh}) & fh \equiv 3 \pmod 8,\ g-\text{even} \\ ((g/2)\sqrt{fh}) & fh \equiv 7 \pmod 8,\ g-\text{even} \\ \mathfrak{p}^2(g\sqrt{fh}) & fh \equiv 1 \pmod 8,\ g-\text{odd} \end{cases}$$

$$\Delta_{K_2/k_2} = \begin{cases} (2^2 g\sqrt{-fh}) & fh \equiv 0 \pmod 2,\ g-\text{odd} \\ (2^3 g\sqrt{-fh}) & fh \equiv 3 \pmod 4,\ g-\text{odd} \\ (2g\sqrt{-fh}) & fh \equiv 3 \pmod 8 \text{ or } fh \equiv 1 \pmod 4,\ g-\text{even};\ fh \equiv 5 \pmod 8,\ g-\text{odd} \\ (\hat{\mathfrak{p}}^2((g/2)\sqrt{-fh}) & fh \equiv 7 \pmod 8,\ g-\text{even} \\ (g\sqrt{-fh}) & fh \equiv 1 \pmod 8,\ g-\text{odd} \end{cases}$$

$$\Delta_{L/K_1} = \begin{cases} (1) & fh \equiv 3 \pmod 4 \\ P^2 & fh = 2 \pmod 4 \\ (P_1 P_2)^2 & fh \equiv 1 \pmod 8,\ g-\text{odd} \\ (2) & \text{otherwise} \end{cases}$$

$$\Delta_{L/K_2} = \begin{cases} (1) & fh \equiv 1 \pmod 4 \\ \hat{P}^2 & fh \equiv 2 \pmod 4 \\ (\hat{P}_1\hat{P}_2)^2 & fh \equiv 7 \pmod 8,\ g-\text{even} \\ (2) & \text{otherwise} \end{cases}$$

$$\Delta_{L/K_3} = \begin{cases} ((1+i)^3 g\sqrt{fh}) & fh \equiv 2 \pmod 4,\ g-\text{odd} \\ (2^2 g\sqrt{fh}) & fh \equiv 3 \pmod 4,\ g-\text{odd} \\ (g\sqrt{fh}) & fh \equiv 3 \pmod 4,\ g-\text{even or } fh \equiv 1 \pmod 8,\ g-\text{odd} \\ ((g/2)\sqrt{fh}) & fh \equiv 7 \pmod 8,\ g-\text{even} \\ (2g\sqrt{fh}) & fh \equiv 1 \pmod 4,\ g-\text{even or } fh \equiv 5 \pmod 8,\ g-\text{odd} \end{cases}$$

$$\Delta_{L/K_3} = \begin{cases} (2^5 f^3 g^2 h^3) & fh \equiv 2 \pmod 4,\ g-\text{odd} \\ (2^4 f^3 g^2 h^3) & fh \equiv 3 \pmod 4,\ g-\text{odd} \\ (f^3 g^2 h^3) & fh \equiv 3 \pmod 8,\ g-\text{even or } fh \equiv 1 \pmod 8,\ g-\text{odd} \\ ((1/4) f^3 g^2 h^3) & fh \equiv 7 \pmod 8,\ g-\text{even} \\ (2^2 f^3 g^2 h^3) & fh \equiv 1 \pmod 4,\ g-\text{even or } fh \equiv 5 \pmod 8,\ g-\text{odd}. \end{cases}$$

PROOF : Integral bases for K_1/Q and K_2/Q are given in Ljunggren[6] or Funakura[3], so the discriminants of these extensions are easily computed. Using the formula

$$\Delta_{K_i/Q} = N_{k_i/Q}(\Delta_{K_i/k_i})\,\Delta^2_{k_i/Q}$$

and the factorizations of (2) given by Parry[8], the values of Δ_{K_i/k_i}, are readily determined.

Since 1, i form a basis for the extension L/K_j it follows that $\Delta_{L/K_j} \mid (2)^2$ for $j = 1, 2$. When $fh \equiv 3 \pmod 4$, no prime divisor of (2) in K_1 ramifies in L, so $\Delta_{L/K_1} = (1)$. Similarly, $\Delta_{L/K_2} = (1)$ when $fh \equiv 1 \pmod 4$.

Since $\Delta_{L/Q} = N_{K_j/Q}(\Delta_{L/K_j}) \cdot \Delta^2_{K_j/Q}$ it follows that

$$\Delta_{L/Q} = \begin{cases} \Delta^2_{K_1/Q} & fh \equiv 3 \pmod 4 \\ \Delta^2_{K_2/Q} & fh \equiv 1 \pmod 4. \end{cases}$$

Therefore $N_{K_1/Q}(\Delta_{L/K_1}) = 2^4$ when $fh \equiv 1 \pmod 4$ and $N_{K_2/Q}(\Delta_{L/K_2}) = 2^4$ when $fh \equiv 3 \pmod 4$.

Using Hilbert's formula for the different it follows that all prime divisors of 2 in K_1 which ramify in L must divide Δ_{L/K_1} to the same exponent. Using this fact and the factorization of (2) given by Parry[8] we have that $\Delta_{L/K_1} = (2)$ if $fh \equiv 1 \pmod 4$ and g is even, or $fh \equiv 5 \pmod 8$ and g is odd, and $\Delta_{L/K_1} = (P_1P_2)^2$ when $fh \equiv 1 \pmod 8$ and g is odd. Similarly, $\Delta_{L/K_2} = (2)$ if $fh \equiv 3 \pmod 4$ and g is odd or if $fh \equiv 3 \pmod 8$ and g is even, $\Delta_{L/K_2} = (\hat{P}_1\hat{P}_2)^2$ when $fh \equiv 7 \pmod 8$ and g is even.

We now show $\Delta_{L/K_1} = P^2$ when $fh \equiv 2 \pmod 4$. We have that $(\sqrt[4]{fg^2h^3} + \sqrt[4]{f^3g^2h})^2 + gh\sqrt{fh} = 2gh\sqrt{fh} + 2fgh + fg\sqrt{fh} \equiv 4\sqrt{fh} \equiv 0 \pmod 4$ so

$(\sqrt[4]{fg^2h^3} + \sqrt[4]{f^3g^2h}) + i\sqrt[4]{fg^2h^3} \equiv 0 \pmod 2$. Thus

$$\left(\frac{\sqrt[4]{fg^3h^3} + \sqrt[4]{f^3g^2h} + i\sqrt[4]{fg^2h^3}}{2}\right)$$

is an integer of L. Now,

$$\Delta_{L/K_1}\left(1, \frac{\sqrt[4]{fg^2h^3} + \sqrt[4]{f^3g^2h} + i\sqrt[4]{fg^2h^3}}{2}\right) = -gh\sqrt{fh}.$$

Since $(gcd((2)^2, (-gh\sqrt{fh}))) = P^2$, $\Delta_{L/K_1} | P^2$. But Hilbert's formula for the different implies $P^2 | \Delta_{L/K_1}$ so $\Delta_{L/K_1} = P^2$. Using a similar argument it follows that $\Delta_{L/K_2} = \hat{P}^2$ when $fh \equiv 2 \pmod 4$.

To compute Δ_{L/K_3}, we use the formula

$$N_{K_3/Q}(\Delta_{L/K_3}) \cdot \Delta^2_{K_3/Q} = \Delta_{L/Q} = N_{K_1/Q}(\Delta_{L/K_1}) \cdot \Delta^2_{K_1/Q}$$

and the values of

$$\Delta_{K_3/Q} = \begin{cases} 2^4 f^2 h^2 & fh \not\equiv 2 \pmod 4 \\ 2^6 f^2 h^2 & fh \equiv 2 \pmod 4 \end{cases}$$

given by Bird and Parry[2]. Since Hilbert's formula for the different shows that all prime divisors of the same rational prime must divide Δ_{L/K_3}, to the same exponent,

$$\Delta_{L/K_3} = N_{K_3/Q}(\Delta_{L/K_1})^{1/4} \Delta^{1/2}_{K_1/Q} / \Delta^{1/2}_{K_3/Q}.$$

The values for Δ_{L/K_3} follow from the formula

$$\Delta_{L/k_3} = N_{K_3/k_3}(\Delta_{L/K_3}) \cdot \Delta^2_{K_3/k_3}$$

and the values of

$$\Delta_{K_3/k_3} = \begin{cases} (fh) & fh \not\equiv 2 \pmod 4 \\ (2fh) & fh \equiv 2 \pmod 4 \end{cases}$$

given by Bird and Parry[2].

4. Integral Bases

In this section necessary and sufficient conditions for the existence of relative integral basis for the quadratic extensions

$$K_1/k_1,\ K_2/k_2,\ L/K_1,\ L/K_2,\ L/K_3$$

and the quartic extension L/k_3 will be given. In addition an integral basis is given for each extension when it exists.

Theorem 2 – Necessary and sufficient conditions are given for the existence of a relative integral basis for the given extension. In addition, an integral basis is given, when it exists. (Note : Here p.d. means principal divisor).

Extension Condition	K_1/k_1	K_2/k_2	L/K_1	L/K_2	L/K_3
$fh \equiv 2 \pmod 4$ g – odd	$\pm h$ is a p.d. of k_1 $1, \theta$	$f = 2$ $1, \bar{\alpha}/2$	$P = (\pi)$ $1, \left(\frac{1+\bar{\eta}+i}{2}\right)\pi$	$\hat{P} = (\hat{\pi})$ $1, \left(\frac{1+\bar{\alpha}+i}{2}\right)\hat{\pi}$	$fh = 2$ $1, \frac{\sqrt{(1+i)g\sqrt{2}(1+\sqrt{2})}}{\sqrt{2}}$
$fh \equiv 3 \pmod 4$ g – odd	$\pm h$ is a p.d. of k_1 $1, \theta$	$h = 1$ $1, \alpha$	Always $1, \frac{\bar{\eta}+i}{2}$	$B_2 = (\beta_2)$ $1, \left(\frac{1+i}{2}\right)\beta_2$	$\pm h$ or $\pm 2h$ is a p.d. of k_1 $1, \theta$
$fh \equiv 3 \pmod 8$ g – even	$\pm h$ is a p.d. of k_1 $1, \delta$	$h = 1$ $1, \alpha$	Always $1, \frac{\bar{\eta}+i}{2}$	$B_2 = (\beta_2)$ $1, \left(\frac{1+i}{2}\right)\beta_2$	$\pm h$ or $\pm 2h$ is a p.d. of k_1 $1, \delta$
$fh \equiv 7 \pmod 8$ g – even	$\pm 2h$ is a p.d. of k_1 $1, \gamma$	$fh = 7$ or $fh = 15$ with $h = 3$ or 5 $1, \varphi$	Always $1, \frac{\bar{\eta}+i}{2}$	$P_1 P_2 = (\hat{\pi})$ $1, \left(\frac{\alpha+i}{2}\right)\hat{\pi}$	$\pm h$ or $\pm 2h$ is a p.d. of k_1 $1, \gamma$
$fh \equiv 1 \pmod 4$ g – even	$\pm h$ is a p.d. of k_1 $1, \theta$	$h = 1$ $1, \alpha$	$B_1 = (\beta_1)$ $1, \left(\frac{1+i}{2}\right)\beta_1$	Always $1, \frac{\bar{\alpha}+i}{2}$	$\pm h$ is a p.d. of k_1 $1, \frac{\theta}{1+i}$
$fh \equiv 5 \pmod 8$ g – odd	$\pm h$ is a p.d. of k_1 $1, \theta$	$h = 1$ $1, \left(\frac{1+\alpha+\bar{\alpha}}{2}\right)$	$B_1 = (\beta_1)$ $1, \left(\frac{1+i}{2}\right)\beta_1$	Always $1, \frac{\bar{\alpha}+i}{2}$	$\pm h$ is a p.d. of k_1 $1, \psi$
$fh \equiv 1 \pmod 8$ g – odd	$I_{\cdot P2} = (\zeta)$ $1, \frac{h+\eta}{\zeta}$	Never	$P_1 P_2 = (\pi)$ $1, \left(\frac{\eta+i}{2}\right)\pi$	Always $1, \frac{\bar{\alpha}+i}{2}$	$\pm h$ is a p.d. of k_1 $1, \zeta$

First, we need :

Lemma – Let $fh \equiv 3 \pmod 4$, $\epsilon_0 = a + b\sqrt{fh}$ and $dh = \epsilon_0 s^2$ with $d = 1$ or 2 and $s \in k_1$. If $d = 1$ then a is odd and $b \equiv 0 \pmod 4$. If $d = 2$ then α is even and b is odd. In addition, if $fh \equiv 7 \pmod 8$ then $a \equiv 0 \pmod 4$.

Proof : Since $fh \equiv 3 \pmod 4$, $a, b \in \mathbb{Z}$ and $s = u + v\sqrt{fh}$ with $u, v, \in \mathbb{Z}$. Thus

$$dh = \epsilon_0 s^2 = (a + b\sqrt{fh})(u + v\sqrt{fh})^2$$

$$= [a(u^2 + v^2 fh) + 2buvfh] + [b(u^2 + v^2 fh) + 2auv]\sqrt{fh}.$$

Since h is odd, a is odd when $d = 1$. Since $fh \equiv 3 \pmod 4$, $1 = a^2 - fhb^2 \equiv 1 - fhb^2 \pmod 8$ so $fhb^2 \equiv 0 \pmod 8$. Thus $b \equiv 0 \pmod 4$. Suppose now $d = 2$

and note $a \not\equiv b \pmod 2$ when $fh \equiv 3 \pmod 4$. If b is even, then a is odd so $2 \equiv dh \equiv a(u^2 + v^2 fh) \equiv u^2 - v^2 \pmod 4$. Since the last congruence has no solution, a is even and b is odd. If $fh \equiv 7 \pmod 8$ then

$$1 = a^2 - fhb^2 \equiv a^2 + 1 \pmod 8$$

so $a^2 \equiv 0 \pmod 8$. Thus $a \equiv 0 \pmod 4$.

Proof of Theorem : First we use the criteria of Mann[7] to determine necessary and sufficient conditions for the existence of an integral basis. In order to determine an integral basis when it exists, it is sufficient to give a basis consisting of integers which has discriminant equal to the field discriminant given by Theorem 1. Since the discriminant of a basis is routinely calculated, it suffices to show the basis consists of integers in each case.

Using Mann's K_1/k_1 has an integral basis if and only if $\Delta_{K_1/k_1} = (D)$ is principal and $K_1 = k_1(\sqrt{D})$ for some generator D of Δ_{K_1/k_1}. From Theorem 1, this is equivalent to $dh = \epsilon s^2$ where $d = g\sqrt{fh}/D$, ϵ is a unit of k_1 and $s \in k_1$. By absorbing square factors into s, unless $fh \equiv 1 \pmod 8$ with g odd, we may take $d \equiv 1$ or 2. Taking norms gives, $(dh)^2 = (s\bar{s})^2$. Thus $\pm dh = s\bar{s}$, so $\pm dh$ is a principal divisor of k_1.

Conversely, if $\pm dh$ is a principal divisor of k_1, then $(dh) = (s)(\bar{s})$ for some $s \in k_1$. Since dh divides $\Delta_{k_1/Q}$, all prime divisors of dh ramify in k_1. Thus $(s) = (\bar{s})$, so $(dh) = (s^2)$. Hence $dh = \epsilon s^2$ for some unit ϵ of k_1 and so K_1/k_1 has an integral basis.

If $dh \neq 1$ then it is not a square in k_1, so $dh = s^2\epsilon$ if and only if ϵ is an odd power of the fundamental unit. Thus we may choose $\epsilon = \epsilon_0$.

If g is odd and $fh \not\equiv 1 \pmod 8$ or if $fh \equiv 1 \pmod 4$ and g is even then $1, \theta$ is an integral basis for K_1/k_1.

If $fh \equiv 3 \pmod 8$ and g is even then $d = 1$, so by the Lemma a is odd and $b \equiv 0 \pmod 4$. If $\pm h$ is a principal divisor, then

$$\theta^2 \equiv g\sqrt{fh}\epsilon_0 \equiv ag\sqrt{fh} \equiv (1 + fh + g\sqrt{fh}) \equiv (1 + \bar{\eta})^2 \pmod 4.$$

Hence $(1+\bar{\eta}+\theta)/2$ is an integer of K_1 and $1, (1+\bar{\eta}+\theta)/2$ is an integral basis for K_1/k_1.

When $fh \equiv 7 \pmod 8$ and g is even then K_1/k_1 has an integral basis if and only if $\pm 2h$ is a principal divisor of k_1. From the Lemma b is odd and $a \equiv 0 \pmod 4$. Thus, if $\pm 2h$ is a principal divisor of k_1, then

$$(g/2)\epsilon_0\sqrt{fh} \equiv (-g/2)b \equiv \left(\frac{(g/2)b - 1}{2}\right)^2 - \left(\frac{(g/2)b+1}{2}\right)^2$$

$$\equiv \left(\frac{(g/2)b - 1}{2}\right)^2 - \left(\frac{(g/2)b+1}{2}\right)^2 + \left(\frac{((g/2)b)^2 - 1}{2}\right)\sqrt{fh}$$

$$\equiv \left[\left(\frac{(g/2)b - 1}{2}\right) + \left(\frac{(g/2)b + 1}{2}\right)\sqrt{fh}\right]^2 \pmod 4.$$

Thus $\sqrt[4]{f(g/2)^2h\epsilon_0^2} \equiv \frac{(g/2)\ b-1}{2} + \frac{(g/2)b+1}{2}\sqrt{fh} \pmod 2$, so

$$\gamma = \frac{(g/2)\ b-1+((g/2)\ b+1)\ \bar{\eta}+2\sqrt[4]{f(g/2)^2h\epsilon_0^2}}{4}$$

is an integer of K_1. Moreover 1, γ is an integral basis for K_1/k_1.

When $fh \equiv 1 \pmod 8$ and g is odd K_1/k_1 has an integral basis if and only if $\mathfrak{p}^2$ is principal and $h = \lambda \epsilon s^2$ where $\mathfrak{p}^2 = (\lambda)$ and $s \in k_1$. The last statement is equivalent to the statement $I_{\mathfrak{p}}$ is principal. Now $(2) = \mathfrak{q}_1\mathfrak{q}_2$ in k_1 where

$$\mathfrak{q}_1 = \left(\frac{h+g\sqrt{fh}}{2}, 2\right) \text{ and } \mathfrak{q}_2 = \left(\frac{h-g\sqrt{fh}}{2}, 2\right).$$

Note $\mathfrak{p} = \mathfrak{q}_1$ or $\mathfrak{p} = \mathfrak{p}_2$. Since $\mathfrak{p}^2 \sim 1 \sim \mathfrak{q}_1\mathfrak{q}_2$, $\mathfrak{q}_1 \sim \mathfrak{p} \sim \mathfrak{q}_2$. Thus $I_{\mathfrak{p}}$ is principal if and only if $I_{\mathfrak{q}_2}$ is principal, say $I_{\mathfrak{q}_2} = (\zeta)$. To show that

$$\frac{h+\sqrt{gh}\sqrt{fh}}{\zeta}$$

is an integer it suffices to show its relative trace and norm for K_1/k_1 are integers. Since the trace, $2h/\zeta$, is in the ideal $I_{\mathfrak{q}_1}$ it is an integer. Now

$$\zeta^2 N_{K_1/k_1}\left(\frac{h+\sqrt{gh}\sqrt{fh}}{\zeta}\right) = h(h-g(\sqrt{fh}) \in I^2\mathfrak{q}_2^2 = (\zeta)^2$$

so the norm is an integer of k_1. Since

$$\mathfrak{p}^2(g\sqrt{fh}) = \Delta_{K_1/k_1}|\Delta_{K_1/k_1}\left(1, \frac{h+\sqrt{gh}\sqrt{fh}}{\zeta}\right) = \left(\frac{4gh\sqrt{fh}}{\zeta}\right) = \mathfrak{q}_1^2(g\sqrt{fh})$$

if follows that $p = q_1$ and that

$$1, \frac{h+\sqrt{gh}\sqrt{fh}}{\zeta}$$

is an integral basis for K_1/k_1.

Next we consider K_2/k_2. Note that $K_2 = Q((1+i)\sqrt[4]{fg^2h^3}) = k_2(\sqrt{2gh}\sqrt{-fh})$. From Mann's criteria and Theorem 1, it follows that K_2/k_2 has an integral basis if and only if $dh = \pm s^2$ for some $s \in k_2$ where $(d) = (2g\sqrt{-fh})/\Delta_{K_2/k_2}$. By absorbing square factors into s we have $d = 2$, 1, or $\hat{\mathfrak{p}}_2^2$. If fh is even or $fh \equiv 1 \pmod 8$ with g odd the equation becomes $2h = \pm s^2$. Hence $f = 2$. In the remaining cases with fh odd, except when $fh \equiv 7 \pmod 8$ and g is even, the equation becomes $h = \pm s^2$, so $h = 1$. When $fh \equiv 7 \pmod 8$ and g is even the condition becomes $\hat{\mathfrak{p}} \sim I$ and $\hat{\mathfrak{p}}^2 = (\sigma)$ is principal. Since $N_{k_2/Q}(\sigma) = 4$, this can happen only when $fh = 7$ or 15. Moreover, when $fh = 15$, $h = 3$ or 5.

When an integral basis exists, it is given by

$$1, \sqrt{-g\sqrt{-2h}} \quad \text{if } f = 2$$

$1, \sqrt{2g\sqrt{-f}}$ if $fh \equiv 3 \pmod 4$ and g is odd

$1, \sqrt{(g/2)\sqrt{-f}}$ if g is even and $fh \equiv 3 \pmod 8$ or $fh \equiv 1 \pmod 4$

$$1, \frac{1 + \sqrt{2g\sqrt{-f}} + \sqrt{-f}}{2} \quad \text{if } fh \equiv 5 \pmod 8 \text{ and } g \text{ is odd}$$

$$1, \frac{1 \pm \sqrt{-7} + \frac{(i+1)}{2}\sqrt{g\sqrt{7}} \pm \frac{(i-1)}{2}\sqrt{7g\sqrt{7}}}{4} \quad \text{if } fh = 7 \text{ and } g \text{ is even}$$

$$1, \frac{h \pm \sqrt{-15} + \frac{(i+1)}{2}\sqrt{gh\sqrt{15}} \pm \frac{(i-1)}{2}\sqrt{15g/h\sqrt{15}}}{4} \quad \text{if } fh = 15 \text{ and } g \text{ is even}$$

where the $+$ holds if and only if $g/2 \equiv 1 \pmod 4$.

We next consider L/K_1. When $fh \equiv 3 \pmod 4$ $\Delta_{L/K_1} = (1)$. Since $L = K_1(\sqrt{-1})$, Mann's criteria holds so L/K_1 has an integral basis. It is easily checked that

$$1, \frac{\sqrt{fh}+i}{2}$$

is an integral basis.

When $fh \equiv 2 \pmod 4$ an integral basis exists if and only if P^2 is principal, say $P^2 = (\tau)$ where $\tau \in K_1$, and $L = K_1(\sqrt{\tau\epsilon})$ where ϵ is a unit of K_1. The second condition is true if and only if $(-1)\cdot s^2 = \tau\cdot\epsilon$ for some $s \in K_1$. Equivalently $(s)^2 = (\tau) = P^2$ or in other words P is a principal ideal in K_1.

When P is principal, say $P = (\tau)$ we have

$$T_{L/K_1}\left(\left(\frac{1+\sqrt{fh}+i}{2}\right)\pi\right) = (1 + \sqrt{fh})\,\pi$$

and

$$N_{L/K_1}\left(\left(\frac{1+\sqrt{fh}+i}{2}\right)\pi\right) = \frac{2+fh}{4}\cdot\pi^2 + \frac{^2\sqrt{fh}\cdot\pi^2}{4}$$

are integers of K_1 since $(\pi)^4 = 2$.

When $fh \equiv 1 \pmod 8$ and g is odd, using Mann's criteria, L/K_1 has an integral basis if and only if $(P_1P_2)^2$ is principal, say $(P_1P_2)^2 = (\tau)$, and $L = K_1(\sqrt{\tau\epsilon})$ where ϵ is a unit of K_1. As in the previous case this is true if and only if P_1P_2 is principal, say $P_1P_2 = (\pi)$. Now

$$T_{L/K_1}\left(\left(\frac{^4\sqrt{fg^2h^3}+i}{2}\right)\pi\right) = {}^4\sqrt{fg^2h^3}\cdot\pi$$

π is an integer of K_1 and

$$N_{L/K_1}\left(\left(\frac{\sqrt[4]{fg^2h^3}+i}{2}\right)\pi\right) = \left(\frac{1+gh\sqrt{fh}}{4}\right)\pi^2.$$

While determining the basis for K_1/k_1 we saw that

$$\mathfrak{p} = \left(\frac{h+g\sqrt{fh}}{2}, 2\right).$$

It follows that

$$1 + gh\bar{\eta} \in \mathfrak{p}^2 = \left(\frac{4}{\pi^2}\right). \quad \text{So} \quad \left(\frac{1+gh\bar{\eta}}{4}\right)\cdot\pi^2$$

is an integer of K_1. Hence,

$$\left(\frac{\sqrt[4]{fg^2h^3}+i}{2}\right)\cdot\pi$$

is an integer of L.

When $fh \equiv 5 \pmod 8$ and g is odd or $fh \equiv 1 \pmod 4$ and g is even, using Mann's criteria, it follows that L/K_1 has an integral basis if and only if B_1 is principal, say $B_1 = (\beta_1)$. Now

$$T_{L/K_1}\left(\left(\frac{1+i}{2}\right)\beta_1\right) = \beta_1 \text{ and } N_{L/K_1}\left(\left(\frac{1+i}{2}\right)\beta_1\right) = \epsilon$$

are integers of K_1 where ϵ is some unit of K_1. So

$$\left(\frac{1+i}{2}\right)\beta_1$$

is an integer of L.

Proofs similar to those for L/K_1 give the results for L/K_2.

We now consider L/K_3. Mann's criteria shows that L/K_3 has an integral basis if and only if $L = K_3(\sqrt{gh}\sqrt{fh}) = K_3(\sqrt{dg}\sqrt{fh}\,\epsilon)$ where $d = (\Delta_{L/K_3}/g\sqrt{fh})$ is determined by Theorem 1 and ϵ is a unit of K_3. This condition is equivalent to $dh = \epsilon\cdot s^2$ for some $s \in K_3$. By absorbing square factors of d into s, we may assume $d = 1 + i$ when $fh \equiv 2 \pmod 4$ and $d = 1$ otherwise.

When $f = 2$ and $h = 1$

$$dh = (1+i) = (1+\sqrt{2})\cdot\left(\frac{\sqrt{2}(1+i)}{1+\sqrt{2}+i}\right)^2 = \epsilon\cdot s^2 \text{ so } L/K_3$$

has an integral basis. Since

$$\omega^4 = \left(\frac{\sqrt{(1+i)}\,g\sqrt{2}(1+\sqrt{2})}{\sqrt{2}}\right)^4 = ig^2(1+\sqrt{2})^2$$

is an integer and 1, ω is a basis for L/K_3 with discriminant generating the field discriminant, it follows that 1,ω form an integral basis for L/K_3.

When $fh \equiv 2 \pmod 4$, $(2)(h)^2 = (s\bar{s})^2$ so (2) is the square of a principal ideal of k_1. Thus, when $fh \neq 2$, Satz 13 of Kuroda[5] applies to show that

$$\epsilon = i^c \left((1 + i) \sqrt{\frac{\epsilon_0}{2}} \right)^j$$

for some integers c and j. By combining square terms, we may take $c, j \in \{0, 1\}$. If $j = 0$ then $(1 + i)h = i^c s^2$ so that $2h^2 = (s\bar{s})^2$. Hence $\sqrt{2} \in k_1$ contradicting that $fh \neq 2$. Thus

$$j = 1 \text{ so } h = i^c \sqrt{\frac{\epsilon_0}{2}}\, s^2.$$

Taking conjugates gives

$$(-i)^c \sqrt{\frac{\epsilon_0}{2}}\, \bar{s}^2 = i^c \sqrt{\frac{\epsilon_0}{2}}\, s^2$$

so $\bar{s}^2 = (-1)^c s^2$.

If $c = 0$ then $s^2 \in k_1$ so either s or $i \cdot s \in k_1$. But $s^2 > 0$ so $s \in k_1$. But $h = \sqrt{\epsilon_0/2} \cdot s^2$, so by squaring and taking norms we obtain $4h^4 = (ss')^4$, a contradiction. If $c = 1$ then $\bar{s}^2 = -s^2$ so $s = t(1 \pm i)$ for some $t \in k_1$. So $h = \pm 2\sqrt{\epsilon_0/2}\, t^2$. Squaring and taking norms again gives

$$h^4 = 4(tt')^4.$$

Thus no solution exists when $fh \equiv 2 \pmod 4$ and $fh \neq 2$.

When $fh \not\equiv 2 \pmod 4$, $d = 1$ so the condition for L/K_3 to have an integral basis becomes $h = \epsilon s^2$. When $fh \neq 3$ either $\epsilon = i^c \epsilon'$ or $\epsilon = i^c(1 + i)\sqrt{\epsilon_0/2}$ where ϵ' is a unit of k_1 and $c = 0$ or 1. If $\epsilon = i^c(1+i)\sqrt{\epsilon_0/2}\, h = (-i)^c(\bar{s})^2\,(1 - i)\sqrt{\epsilon_0/2}$. Thus $s^2\,(1+i) = (i^{-2c})\,\bar{s}^2\,(1 - i) = (i^{-2c})\,\bar{s}^2\,(1+i)\,(-i)$. So $(\frac{s}{\bar{s}})^2 = (i^{1-2c})$. This equation has no solution in K_3.

If $\epsilon = \epsilon'$ then $h/\epsilon = s^2 \in k_1$. If $s \in k_1$ it follows that $\pm h$ is a principal divisor of k_1. If $s \notin k_1$ we have $k_1(s) = K_3 = k_1(\sqrt{-1})$. So $h/\epsilon = -t^2$ for $t \in k_1$. Again it follows that $\pm h$ is a principal divisor of k_1. On the other hand if $\pm h$ is a principal divisor of k_1 then $h = s^2\epsilon$ for some ϵ, $s \in k_1$ where ϵ is a unit.

If $\epsilon = i\epsilon'$ we have $h = s^2 i\epsilon'$ so $h = -\bar{s}^2 i\epsilon'$. Hence $s^2 = -\bar{s}^2 = (i\bar{s})^2$ and so $s = \pm i\bar{s}$. This implies $s = x(1 \pm i)$ for some $x \in k_1$. Then $s^2 = \pm 2x^2 i$ and so $\pm h = 2x^2\epsilon'$. Therefore $\pm 2h = (2x)^2\epsilon'$. It follows that $\pm 2h$ is a principal divisor of k_1. Note that since $2h$ and $-2h$ are not squares in k_1 we may choose $\epsilon' = \epsilon_0$. On the other hand when $\pm 2h$ is a principal divisor of k_1 it follows that $\pm 2h = x^2\epsilon_0$ for some $x \in k_1$. Therefore $\pm h = (x/(1+i))^2\,(i\epsilon_0)$. Note that the equation $\pm 2h = x^2\epsilon_0$ does not have a solution for $x \in k_1$ when $fh \equiv 1 \pmod 4$, since then 2 is unramified in k_1.

When $fh = 3$, $h = 1 = 1 \cdot 1^2$. Thus in this case L/K_3 always has an integral basis and h is a principal divisor of k_1.

When $\pm h$ is a principal divisor of k_1 we have seen that $\theta = \sqrt[4]{fg^2h\,\epsilon_0^2} \in L$.

When $\pm 2h$ is a principal divisor of k_1, we have $\pm 2h = s^2 \epsilon_0$ for some $s \in k_1$. Hence

$$h^2 = \left(\frac{s}{1+i}\right)^4 (-\epsilon_0^2).$$

Thus

$$\sqrt[4]{fg^2h^3} = \frac{s}{1+i}\sqrt[4]{-fg^2h\epsilon_0^2} = \frac{s}{1+i}\cdot\theta.$$

So it follows that $\theta \in L$.

It was shown while determining the basis for K_1/k_1 that when $fh \equiv 3 \pmod 8$, g is even, and $\pm h$ is a principal divisor of k_1

$$\delta = \frac{1+\bar{\eta}+\theta}{2}$$

is an integer of L. If $\pm 2h$ is a principal divisor of k_1, then by the Lemma a is even and b is odd. Thus, $(1+i)^2 = 2i \equiv gbf\,hi + ag\sqrt{-fh} \equiv g\epsilon_0\sqrt{-fh} \pmod 4$. So $1 + i \equiv \sqrt[4]{-fgh\epsilon_0^2} \pmod 2$ and

$$\delta = \frac{1+i+\sqrt[4]{-fg^2h\epsilon_0^2}}{2}$$

is an integer of L.

While determining the basis for K_1/k_1 it was shown when $fh \equiv 7 \pmod 8$, g is even, and $\pm 2h$ is a principal divisor of k_1 that γ is an integer of L. When

$$h = 1, \left(\frac{g/2-\sqrt{f}}{1+i}\right)^2 \equiv (g/2)i\sqrt{f} \equiv (\sqrt[4]{-f(g/2)^2})^2 \pmod 4.$$

Thus
$$\gamma = \frac{\left(\frac{g/2-\sqrt{f}}{1+i}\right) + \sqrt[4]{-f(g/2)^2}}{2} = \frac{(g/2-\sqrt{f})(1-i)+2\sqrt[4]{-f(g/2)^2}}{4}$$

is an integer of L. When $h \neq 1$ and $\pm h$ is a principal divisor of k_1, since a is odd and

$$b \equiv 0 \pmod 4, \left(\frac{ag/2-\sqrt{fh}}{1+i}\right)^2 \equiv (ag/2)\,i\sqrt{fh}$$
$$\equiv (\sqrt[4]{-f(g/2)^2 h\epsilon_0^2}) \pmod 4.$$

Hence
$$\gamma = \frac{\left(\frac{a\cdot g/2-\sqrt{fh}}{1+i}\right) + \sqrt[4]{-f(g/2)^2 h\epsilon_0^2}}{2}$$
$$= \frac{(a\cdot g/2-\sqrt{fh})(1-i) + 2\sqrt[4]{-f(g/2)^2 h\epsilon_0^2}}{4}$$

is an integer of L.

When $fh \equiv 1 \pmod 4$, g is even, and $\pm h$ is a principal divisor of k_1,

$$\left(\frac{\theta}{1+i}\right)^2 = (-g/2)\, i \cdot \sqrt{fh} \cdot \epsilon$$

where $\epsilon = 1$ if $h = 1$ and $\epsilon = \epsilon_0$ otherwise, which is an integer of L.

When $fh \equiv 5 \pmod 8$, g is odd and $\pm h$ is a principal divisor of k_1 it follows that $N(\epsilon_0) = 1$. Note that $\epsilon_0^3 = r + s\sqrt{fh}$ where $r, s \in \mathbb{Z}$ with r odd and $s \equiv 0 \pmod 4$. Now $[(1+i)\,\epsilon_0\theta]^2 \equiv 2i\epsilon_0^2\, g\sqrt{fh} \equiv 2\sqrt{-fh} \equiv 1 - fh + 2\sqrt{-fh} \equiv (1+\bar{\alpha})^2 \pmod 4$. Thus

$$\frac{1 + \bar{\alpha} + (1+i)\,\epsilon_0\theta}{2}$$

is an integer of L.

Finally, we consider the case $fh \equiv 1 \pmod 8$ and g-odd. Let

$$a_0 + b_0\sqrt{fh} = \epsilon = \begin{cases} \epsilon_0 & h \neq 1 \\ 1 & h = 1. \end{cases}$$

Note that since $fh \equiv 1 \pmod 8$, a_0 and b_0 are integers. When $\pm h$ is a principal divisor of k_1, $N(\epsilon) = 1$. Since $a_0^2 - b_0^2\, fh \equiv a_0^2 - b_0^2 \equiv 1 \pmod 8$, a_0 is odd and $b_0 \equiv 0 \pmod 4$. Thus

$$\left(\frac{ga_0 + \sqrt{-fh}}{1+i}\right)^2 \equiv ga_0\sqrt{fh} \equiv (\sqrt[4]{fg^2 h\,\epsilon^2})^2 \pmod 4.$$

Therefore,

$$\frac{(ga_0 + \sqrt{-fh})\,(1-i) + 2\sqrt[4]{fg^2 h\,\epsilon^2}}{4}$$

is an integer of L.

Theorem 3 — L/k_3 always has an integral basis. A basis is given by

$$1,\ \frac{\eta(1+i)+\bar{\bar{\eta}}}{2},\ \frac{\bar{\eta}(1+i)}{2},\ \frac{\bar{\bar{\eta}}\,(1+i)}{2} \qquad fh \equiv 2 \pmod 4,\ g-\text{odd}$$

$$1,\ \eta,\ \frac{1+i\bar{\eta}}{2},\ \frac{\eta+i\bar{\bar{\eta}}}{2} \qquad fh \equiv 3 \pmod 8,\ g-\text{even}$$

$$1,\ \frac{1+i\bar{\eta}}{2},\ \frac{1+\eta+\bar{\eta}}{2},\ \frac{1+3i+\eta+(1+i)\,\bar{\eta}+(-1)^{(h-1)/2}\,i\bar{\bar{\eta}}}{4} \qquad fh \equiv 3 \pmod 8,\ g-\text{even}$$

$$1,\ \frac{1+i\bar{\eta}}{2},\ \frac{1+\eta+(g/2)\,h\bar{\eta}}{2(1+i)},\ \frac{1+(g/2)fi+\eta+((g/2)h+i)\bar{\eta}+(-1)^{(h-1)/2}i\bar{\bar{\eta}}}{4(1+i)} \qquad fh \equiv 7 \pmod 8,\ g-\text{even}$$

$$1,\ \alpha,\ \frac{1+i\bar{\alpha}}{2},\ \frac{\alpha+i\bar{\bar{\alpha}}}{2} \qquad fh \equiv 1 \pmod 4,\ g-\text{even}$$

$$1, \frac{1+i\bar{\alpha}}{2}, \frac{1+\alpha+\bar{\alpha}}{2}, \frac{1+3i+\alpha+(1+i)\bar{\alpha}+(-1)^{(h-1)/2}\, i\overline{\overline{\alpha}}}{4}$$

$fh \equiv 5 \pmod 8$, g – odd

$$1, \frac{1+i\bar{\alpha}}{2}, \frac{1+\alpha+gh\bar{\alpha}}{2(1+i)}, \frac{1-fgi+\alpha+(gh+i)\bar{\alpha}+(-1)^{(h-1)/2}\, i\overline{\overline{\alpha}}}{4(1+i)}$$

$fh \equiv 1 \pmod 8$, g – odd

Proof : Since k_3 has class number 1, L/k_3 always has an integral basis. As in Theorem 2, it is only necessary to check that all elements of each basis are integers and that its discriminant equals the field discriminant in each case. The latter is routine and will be left to the reader.

When $fh \equiv 0 \pmod 2$ and g is odd, $\bar{\eta}^2 = fh \equiv (i\bar{\eta})^2 \pmod 4$. So $\bar{\eta} \equiv i\bar{\eta} \pmod 2$ and thus $((1+i)/2)\,\bar{\eta}$ is an integer. Since

$$\left(\frac{\eta(1+i)+\overline{\overline{\eta}}}{2}\right)^2 = \left(\frac{fg/2+ghi}{2}\right)\bar{\eta} + \frac{fgh\,(1+i)}{2}$$

and

$$fg/2 \equiv gh \equiv 1 \pmod 2, \left(\frac{fg/2+ghi}{2}\right)\bar{\eta}$$

is an integer of L. Hence

$$\frac{\eta(1+i)+\overline{\overline{\eta}}}{2}$$

is an integer. Since $(\overline{\overline{\eta}})^2 = fg\sqrt{fh} \equiv -fg\sqrt{fh} \equiv (i\overline{\overline{\eta}})^2 \pmod 4$, $\overline{\overline{\eta}} \equiv i\overline{\overline{\eta}} \pmod 2$ and so $((1+i)/2)\,\overline{\overline{\eta}}$ is an integer of L.

When $fh \equiv 3 \pmod 4$, $(i\bar{\eta})^2 = -fh \equiv 1 \pmod 4$. So $i\bar{\eta} \equiv 1 \pmod 2$ and therefore $(1+i\bar{\eta})/2$ is an integer of L. We also have that $h \equiv f^2h \equiv -f \pmod 4$ and thus $\eta^2 \equiv gh\sqrt{fh} \equiv -fg\sqrt{fh} \equiv (i\overline{\overline{\eta}})^2 \pmod 4$. Thus $\eta \equiv i\overline{\overline{\eta}} \pmod 2$ so $(\eta + i\overline{\overline{\eta}})/2$ is an integer of L.

When $fh \equiv 3 \pmod 8$ and g is even, $(1+\bar{\eta})^2 = (1+fh)+2\sqrt{fh} \equiv gh\sqrt{fh} \equiv \eta^2 \pmod 4$. Therefore $1+\bar{\eta} \equiv \eta \pmod 2$ and thus $(1+\eta+\bar{\eta})/2$ is an integer of L. Hence

$$\left(\frac{1+i\bar{\eta}}{2}\right)\left(\frac{1+\eta+\bar{\eta}}{2}\right) = (1/4)\,[1+ifh+\eta+(1+i)\,\bar{\eta}+ih\,\overline{\overline{\eta}}]$$

is an integer of L. Since $fh \equiv 3 \pmod 4$, $(1/4)\,[1+3i+\eta+(1+i)\,\bar{\eta} + (-1)^{(h-1)/2}\, i\,\overline{\overline{\eta}}]$ is also an integer of L.

Suppose now that $fh \equiv 7 \pmod 8$ and g is even. Since $\eta^2 = gh\sqrt{fh} \equiv (1+(g/2)h\bar{\eta})^2 \pmod 8$, is follows that

$$\frac{1+\eta+(g/2)h\bar{\eta}}{2(1+i)}$$

is an integer of L. Hence

$$\left(\frac{1+i\bar{\eta}}{2}\right)\left(\frac{1+\eta+(g/2)h\bar{\eta}}{2(1+i)}\right) = \frac{1}{4(1+i)}[1+(g/2)fh^2 i$$

$$+\eta+((g/2)h+i)\bar{\eta}+ih\,\overline{\overline{\eta}}\,]$$

is an integer of L. Also, since g is even $(1+i)\mid \overline{\overline{\eta}}$. Thus

$$\frac{1}{4(1+i)}[(1+(g/2)fi)+\eta+((g/2)h+i)\,\bar{\eta}$$

$$-\left(\frac{h-(-1)^{(h-1)/2}}{4}\right)\left(\frac{i\,\overline{\overline{\eta}}}{1+i}\right)$$

$$= \frac{1}{4(1+i)}[(1+(g/2)fi)+\eta+((g/2)h+i)\bar{\eta}+(-1)\bar{\eta}+(-1)^{(h-1)/2}\,i\,\overline{\overline{\eta}}\,]$$

is an integer of L.

When $fh \equiv 1 \pmod 4$,

$$\frac{1+i\bar{\alpha}}{2} = \frac{1-\bar{\eta}}{2} = \frac{1-\sqrt{fh}}{2}$$

is an integer. In addition $f \equiv fh^2 \equiv h \pmod 4$, so when g is even, $\alpha^2 = (g/2)ih\sqrt{fh} \equiv (g/2) \pmod 4$ if $\sqrt{fh} \equiv (i\bar{\alpha})^2 \pmod 4$. Hence $(\alpha+i\bar{\alpha})/2$ is an integer of L.

When $fh \equiv 5 \pmod 8$ and g is odd then $(1+\bar{\alpha})^2 = (1-fh)+2\sqrt{-fh} \equiv 2gh\sqrt{-fh} \equiv \alpha^2 \pmod 4$. Therefore $(1+\alpha+\bar{\alpha})/2$ is an integer of L. Hence

$$\left(\frac{1+i\bar{\alpha}}{2}\right)\left(\frac{1+\alpha+\bar{\alpha}}{1}\right) = \frac{(1-fhi)+\alpha+(1+i)\bar{\alpha}+hi\overline{\overline{\alpha}}}{4}$$

is an integer of L. Since $fh \equiv 5 \pmod 8$ it follows that

$$\frac{1+3i+\alpha+(1+i)\bar{\alpha}+(-1)^{(h-1)/2}\,i\overline{\overline{\alpha}}\,)}{4}$$

is integer of L.

When $fh \equiv 1 \pmod 8$ and g is odd, $(1+gh\bar{\alpha})^2 = (1-fg^2h^3)+2gh\sqrt{-fh} \equiv 2gh\sqrt{-fh} \equiv \alpha^2 \pmod 8$. Thus

$$\frac{1+\alpha+gh\bar{\alpha}}{2(1+i)}$$

is an integer of L. Hence

$$\left(\frac{1+i\bar{\alpha}}{2}\right)\left(\frac{1+\alpha+gh\bar{\alpha}}{2(1+i)}\right) = \frac{1}{4(1+i)}[(1-fgh^2i)+\alpha+(gh+i)\bar{\alpha}+ih\overline{\overline{\alpha}}]$$

is an integer of L. Since f and g are odd $(1+i)$ divides $\overline{\overline{\alpha}}$. So

$$\frac{1}{4(1+i)}[(1-fgi)+\alpha+(gh+i)\bar{\alpha}+ih\overline{\overline{\alpha}}\,]-\left(\frac{h-(-1)^{(h-1)/2}}{4}\right)\left(\frac{i\,\overline{\overline{\alpha}}}{1+i}\right)$$

$$= \frac{1}{4(1+i)} [(1 - fgi) + \alpha + (gh+i)\bar{\alpha} + (-1)^{(h-1)/2}\, i\bar{\bar{\alpha}}]$$

is an integer of L.

5. Class Number Considerations

In this section we explicitly compute the value of ρ, which we defined in section 2, and show that it is related to the rank of the 2-class group of k_1. Similar results for cyclic quartic fields were obtained by the authors[4]. Corollary 3 of this section is a variation on Theorem 2 of Funakura[3]. Here 1, $-fh$ will be referred to as trivial principal divisors of k_1 while all others will be called nontrivial.

Theorem 4 – If $N(\epsilon_0) = -1$ then

$$\rho = \begin{cases} 0 & \text{if } fh \equiv 1 \pmod 8 \text{ with } g \text{ odd and } \mathfrak{p}_2^2 \neq (1). \\ \dfrac{1}{2^{t-1}} & \text{otherwise.} \end{cases}$$

If $N(\epsilon_0) = +1$ and there are no nontrivial odd principal divisors of k_1, then $\rho = 1/(2^{t-2})$. If $N(\epsilon_0) = +1$ and k_1 has a nontrivial odd principal divisor then

$$\rho = \begin{cases} \dfrac{1}{2^{t-3}} & \text{If } fh \equiv 3 \pmod 8, \text{ or } fh \equiv 7 \pmod 8 \text{ with } g \text{ odd.} \\ \dfrac{1}{2^{t-2}} & \text{If } fh \equiv 2 \pmod 4, fh \equiv 5 \pmod 8, \text{ or } fh \equiv 1 \pmod 8 \text{ and } \mathfrak{p}_2 \\ & \text{generates a strongly ambiguous class of } k_1 \text{ when } g \text{ is odd.} \\ 0 & \text{otherwise.} \end{cases}$$

Proof : To insure $f > 1$, fix a prime divisor q of fh and require it to divide f. When fh is even we will choose $q = 2$.

When $N(\epsilon_0) = -1$ then $fh \equiv 3 \pmod 4$ and Theorem 3.1 of Barrucand and Coha[1] shows that k_1 has no nontrivial principal divisors. If $fh \equiv 1 \pmod 8$ with g odd and $\mathfrak{p}_2^2 \neq (1)$ then it follows from Theorem 2 that K_1/k_1 never has an integral basis. Otherwise, there is exactly one value of h for which K_1/k_1 has an integral basis and exactly 2^{t-1} possible values for h, so $\rho = 1/2^{t-1}$.

Assume now that $N(\epsilon_0) = +1$ and that k_1 has no nontrivial odd principal divisors. Note this can only occur when $fh \equiv 3 \pmod 4$. If g is odd of $fh \equiv 3 \pmod 8$ with g even then Theorem 2 shows that K_1/k_1 has an integral basis exactly when $h = 1$. Now the two nontrivial principal divisors n_1 and n_2 are both even and we may assume q/n_1. Thus when $fh \equiv 7 \pmod 8$ and g is even, K_1/k_1 has an integral basis if and only if $2h = \pm n_2 > 0$. Since there are 2^{t-2} possible values for h, $\rho = 1/2^{t-2}$.

Assume now that $N(\epsilon_0) = +1$ and that k_1 has a nontrivial odd principal divisor. If $fh \equiv 3 \pmod 4$ then both principal divisors n_1 and n_2 must be odd and as above we assume $q | n_1$. If $fh \equiv 3 \pmod 8$ or $fh \equiv 7 \pmod 8$ with g odd then Theorem 2 shows K_1/k_1 has an integral basis if and only if $h = 1$ or $\pm n_2$ with the sign chosen so that $h > 0$. Since there are 2^{t-2} possible values for h, $\rho = 1/(2^{t-3})$.

If $fh \equiv 2 \pmod 4$, $fh \equiv 5 \pmod 8$ or $fh \equiv 1 \pmod 8$ with g even, then as above there are exactly two values of h such that K_1/k_1 has an integral basis. However, here there are 2^{t-1} possible values for h, so $\rho = 1/(2^{t-2})$. Suppose $fh \equiv 1 \pmod 8$ with g odd and $\mathfrak{p}_2$ generates a strongly ambiguous class of k_1. Thus $\mathfrak{p}_2 \sim I$ where I is an ambiguous ideal of k_1. This means $I \mid (\sqrt{fh})$. Since $(\sqrt{fh})$ is a principal ideal of k_1, we can always choose I with $(I, (q)) = 1$. Now the principal divisor (n_2) satisfies $J^2 = (n_2)$ for some ideal J of k_1. Let $IJ = J_1^2 J_2$ where J_2 is a square free ideal of k_1. Then K_1/k_1 has an integral basis precisely for the values $h = h_1$ and $h = h_2$ where $h_1 > 0$, $h_2 > 0$ and $I^2 = (h_1)$, $J_2^2 = (h_2)$. As above $\rho = 1/(2^{t-2})$. If $fh \equiv 1 \pmod 8$ with g odd and $\mathfrak{p}_2$ does not belong to a strongly ambiguous class of k_1, then $\mathfrak{p}_2 \neq I$ or $I\mathfrak{p}_2 \neq (1)$ for any choice of I. Thus no extension K_1/k_1 has an integral basis in this case. If $fh \equiv 7 \pmod 8$ and g is even, then since all principal divisors must be odd, it also follows that no extension K_1/k_1 has an integral basis.

Corollary 1 – Let u denote the rank of the 2-class group of k_1 and assume $\rho \neq 0$. If $N(\epsilon_0) = -1$ then $\rho = 1/2^u$. If $N(\epsilon_0) = +1$ then

$$\rho = \begin{cases} \dfrac{1}{2^{u-1}} & \text{if } fh \text{ is the sum of two squares, or } fh \equiv 3 \pmod 4 \text{ and } k_1 \\ & \text{has a nontrivial odd principal divisor.} \\ \dfrac{1}{2^u} & \text{if } fh \equiv 1, 2 \pmod 4 \text{ and is not the sum of two squares, or} \\ & fh \equiv 3 \pmod 4 \text{ and } k_1 \text{ has no nontrivial odd principal} \\ & \text{divisors.} \end{cases}$$

Proof : It is shown in Theorem 3.1 Barrucand and Cohn[1] that $u = t - 1$ or $t - 2$ according as fh is the sum of two squares or not. The results are now immediate from Theorem 4.

Corollary 2 – If $\rho \neq 0$ and k_1 has odd class number then every extension K_1/k_1 has an integral basis.

Proof : Since k_1 has odd class number $u = 0$. If fh is the sum of two squares then $t = 1$, so $N(\epsilon_0) = -1$ and $\rho = 1$. If $fh \equiv 3 \pmod 4$ then $t = 2$, so no nontrivial odd principal divisors exist. Thus we always have $\rho = 1$, so K_1/k_1 always has an integral basis.

Corollary 3 – For a fixed real quadratic field k_1, K_1/k_1 has an integral basis for all pure quartic fields K_1 containing k_1, if and only if $\rho \neq 0$ and $fh = 2, p, q, 2q, q_1q_2$, or $fh = 2p, p_1p_2$ with $N(\epsilon_0) = +1$, or $fh = pq$ where $\pm p$ and $\pm q$ are principal divisors. Here the p's and q's denote primes which are congruent to 1 and 3 (mod 4) respectively.

Proof : The first condition is equivalent to $\rho = 1$. When $N(\epsilon_0) = -1$, then $t = 1$ so $fh = 2$ or p. Since k_1 has $N(\epsilon_0) = -1$ for these values of fh, the converse also holds. If $fh \equiv 1, 2 \pmod 4$ and is not the sum of two squares or $fh \equiv 3 \pmod 4$ and no nontrivial odd principal divisors exist then $\rho = 1$ if and only if $t = 2$ which is true exactly when $fh = q, 2q$ or q_1q_2. If $N(\epsilon_0) = +1$ and fh is the sum of two squares, $\rho = 1$ if and only if $t = 2$ so $fh = 2p$ or p_1p_2. If $fh \equiv 3 \pmod 4$ and k_1 has nontrivial odd principal divisors, $\rho = 1$ if and only if $t = 3$ so $fh = pq$. Here the nontrivial odd principal divisors must necessarily be $\pm p$ and $\pm q$.

It schould be noted that the requirement $\rho \neq 0$ in Corollaries 2 and 3 cannot be dropped. For example, when $fh = 321 = 3 \cdot 107$ the field k_1 has class number 3 and $\mathfrak{p}_2$ belongs to an ideal class of order 3. hence K_1/k_1 never has an integral basis when g is odd.

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