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A NOTE ON THE SPECTRA OF OPERATOR SCHUR MATRICES AND THE MAXIMAL GROUP OF THE ALGEBRA OF TRIANGULAR CONSERVATIVE MATRICES

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In this paper, some operator matrix classes are first characterized and the closure of the class $(\omega(X), Y)$ is determined where Y is a subspace of $c(Y)$. Then necessary and sufficient conditions are obtained for a matrix (A_{nk}) to lie in the maximal group $G(X)$ of the Banach algebra $\Delta(X)$. These results are used to find necessary and sufficient conditions under which a triangular operator Schur matrix lies in the boundary $\partial G(X)$ of $G(X)$. Lastly, the spectrum $\sigma(A)$ of a triangular operator Schur matrix $A = (A_{nk})$ is determined.

§1. Let X be a complex normed linear space with θ as its zero element and $c(X)$, $c_0(X)$, $l_\infty(X)$, $\omega(X)$ denote respectively the spaces of convergent, null, bounded and all sequences of elements of X. If X is the space of complex numbers, it is customary to write the corresponding spaces as c, c_0, l_∞ and ω respectively. Let X, Y be normed linear spaces and $A = (A_{nk})$ be an infinite matrix of linear operators on X into Y. For any $x = (x_k) \in \omega(X)$, we define the sequence $Ax = (A_n x)$ formally by

$$A_n x = \sum_k A_{nk} x_k.$$

If $E \subset \omega(X)$, $F \subset \omega(Y)$, the set (E, F) is defined as the aggregate of all matrices $A = (A_{nk})$ such that $Ax \in F$ whenever $x \in E$. The set $c_A = \{x \in \omega(X) : Ax \in c(X)\}$ is called the 'convergence field' of $A = (A_{nk})$.

The classical theorems of Kōjima, Schur, Toeplitz give characterizations of the spaces (c, c), (l_∞, c), (l_∞, l_∞) and so on. Characterizations of the spaces $(c(X), c(Y))$, $(l_\infty(X), c(Y))$, $(l_\infty(X), l_\infty(Y))$ are well-known, when X, Y are Banach spaces[2,6].

The crucial concept in the latter characterizations is that of a group norm, introduced by Robinson[6]. If (T_k) is a sequence of bounded linear operators on X into Y, $S(X)$ denotes the closed unit disc in X and N is set of all natural numbers, the 'group norm' $\|(T_k)\|$ is defined as

$$\sup_{\substack{n \in N \\ x_k \in S(X)}} \left| \sum_{k=1}^{n} T_k x_k \right|$$

It is familiar[3,8] that (l_∞, l_∞) is a Banach algebra with identity under matrix product and norm defined by

$$\|A\| = \sup_n \sum_{k=1}^{n} |a_{nk}|.$$

Further, (c, c) and the space Δ of lower triangular matrices in (c, c) are closed subalgebras of (l_∞, l_∞). If G denotes the maximal group of Δ and ∂G is its boundary in Δ, it is easy to see that $A \in G$ if and only if A is normal (that is, A is triangular, having no zeros on the main diagonal) and $c_A = c$. Properties of Δ have been extensively studied by various authors.

The present object is to settle certain problems for the algebra $\Delta(X)$ of lower triangular matrices $A \in (c(X), c(X))$ where each $A_{nk} \in B(X)$, the Banach algebra of bounded linear operators on X. Ramanujan[4] has shown that $\Delta(X)$ is a Banach algebra with unit element under matrix product and norm defined by

$$\|A\|_1 = \sup_{\substack{m,n \in N \\ x_k \in S(X)}} \left\| \sum_{k=1}^{m} A_{nk} x_k \right\| = \sup_n \|(A_{nm})_{m \in N}\|^{\dagger}. \qquad \ldots(1.1)$$

In the present paper, conditions have been obtained for a matrix $A = (A_{nk})$ to lie in the maximal group $G(X)$ of $\Delta(X)$. Secondly, the spectrum $\sigma(A)$ of $A \in (l_\infty(X), c(X)) \cap \Delta(X)$ has been determined. The proofs of these necessitate certain characterization theorems and related results, which have been first established. These, in turn have been used to find necessary and sufficient conditions under which an element of $(l_\infty(X), c(X)) \cap \Delta(X)$ lies in $\partial G(X)$. The scalar analogues of these results were obtained earlier[5,7,8].

Throughout, we take X, Y to be Banach spaces. A matrix $A = (A_{nk})$ is said to be 'right-finite' if there is $n_0 \in N$ such that $A_{nk} = \bar{\theta}$ for all $n \in N$ and all $k \geq k_0$, where $\bar{\theta}$ denotes the zero operator on X. The identity operator on X is written as I. Further, $\mathcal{J}$ is the operator matrix with I everywhere on the main diagonal and $\bar{\theta}$ elsewhere.

§2. The following theorems are established.

Theorem 1 – If F is a subspace of $l_\infty(Y)$, then $A = (A_{nk}) \in (\omega(X), F)$ if and only if

$$(A_{nk} z)_{n \in N} \in F \text{ for each fixed } k \in N \text{ and each } z \in X \qquad \ldots(2.1)$$

and

$$A \text{ is right-finite.} \qquad \ldots(2.2)$$

Theorem 2 – If F is a subspace of $c(Y)$, then every $A \in (l_\infty(X), F)$ is a limit of a sequence of elements in $(\omega(X), F)$ under any norm $\|\cdot\|$ satisfying $\|\cdot\| \leq H\|\cdot\|_1$ for some constant $H > 0$.

[†]The characterization of $(c(X), c(Y))$ is due to Robinson[6]. See, for instance, Maddox[2] (Theorem 4.2), where A_{nk} are not necessarily bounded.

The result is false if $Y = X$ and $F = l_\infty(X)$.

Theorem 3[‡] – If $A = (A_{nk})$ is a lower triangular matrix of linear operators on X into Y, then the necessary and sufficient conditions that there is a lower triangular matrix $X = (X_{nk})$ (necessarily unique) with $AX = \mathcal{J} = XA$ and $X_{nk} \in B(X)$ for all $n, k \in N$, are that

(i) $A_{nk} \in B(X)$ for all $n, k \in N$, and

(ii) for each $n \in N$, A_{nn} is invertible in $B(X)$.

Theorem 4 – A necessary and sufficient condition for $A = (A_{nk}) \in \partial G(X)$ whenever $A \in (l_\infty(X), c(X)) \cap \Delta(X)$ is that for each $n \in N$, A_{nn} lies in the closure of the maximal group of $B(X)$*.

Theorem 5 – If $A = (A_{nk}) \in (l_\infty(X), c(X)) \cap \Delta(X)$ and $\sigma(A)$ denotes the spectrum of A with regard to the Banach algebra $\Delta(X)$, then $\sigma(A) = \bigcup_{i \in N} \sigma(A_{ii}) \cup \{0\}$.

§3. The following lemmas are needed.

Lemma 1 (Ramanujan[4], Corollary) – $\Delta(X)$ is a Banach algebra with unit element $\mathcal{J}$, under matrix product and norm defined by (1.1).

Lemma 2 (Maddox[2], Proposition 4.6) – Let X, Y be Banach spaces and A_{nk}, $n, k \in N$, be bounded linear operators on X into Y. For each $n, m \in N$, write $R_{n,m} = (A_{nm}, A_{n,m+1}, \ldots)$.

Then

$A \in (l_\infty(X), c(Y))$ if and only if

(i) $\lim_n A_{nk} = A_k$ exists for each fixed $k \in N$,

(ii) $\lim_m \|R_{nm}\| = 0$ for each $n \in N$, and

(iii) $\sup_n \|R_{nm} - R_m\| \to 0$ $(m \to \infty)$, where $R_m = (A_m, A_{m+1}, \ldots)$.

When (i), (ii), (iii) hold, we have

$$\lim_n \sum_{k=1}^{\infty} A_{nk} x_k = \sum_{k=1}^{\infty} A_k x_k, \text{ for each } x = (x_k) \in l_\infty(X).$$

It is to be understood in (ii) that $\|R_n\| < \infty$ for each $n \in N$ and the convergence in (*i*) is pointwise.

§4. *Proof of Theorem* 1 – Sufficiency is trivial. The necessity of (2.1) follows by considering for each $z \in X$ the sequence whose kth term is z and other are θ.

To prove the necessity of (2.2), let $A = (A_{nk}) \in (\omega(X), F)$. First we show that A is row-finite. If not, we can find $n \in N$ and an increasing sequence (m_i)

‡For scalar matrices, see Cooke[1] (Theorem 2.1, I).

*This condition is pertiment in view of the fact that the maximal group of $B(X)$ and its boundary do not comprise, in general, of the whole of $B(X)$ (Yood[1]).

of natural numbers such that $A_{nm_i} \neq \bar{\theta}$ for each for each $i \in N$. If $x_{m_i} \in X$ is such that $A_{nm_i} x_{m_i} \neq \bar{\theta}$ for each $i \in N$. If $x_{m_i} \in X$ is such that $A_{nm_i} x_{m_i} \neq \theta$ for each $i \in N$ and $y = (y_k) \in \omega(X)$ is defined by

$$Y_k = \begin{cases} \theta, & k \neq m_i,\ i \in N, \\ \dfrac{x_{m_i}}{\|A_{nm_i} x_{m_i}\|}, & k = m_i,\ i \in N \end{cases}$$

then $A_n y$ does not exist, since

$$\sum_k A_{nk} Y_k = \sum_i \frac{A_{nm_i} x_{m_i}}{\|A_{nm_i} x_{mi}\|}$$

is not convergent. This proves our assertion.

If condition (2.2) is false, then one can find increasing sequences (n_i) and (k_{n_i}) of natural numbers such that

$$A_{n_i k_{n_i}} \neq \bar{\theta} \text{ for each } i \in N. \qquad \ldots(4.1)$$

Since A is row-finite, for each $i \in N$ we can choose k_{n_i} to be the highest suffix for which (4.1) holds. We construct $x = (x_n) \in \omega(X)$ such that $Ax \notin l_\infty(X)$.

For this, we set $x_n = \theta$ if $n \neq k_{n_i}$, $i \in N$ and for $n = k_{n_i}$ let x_n be defined inductively as follows. Let $x_{k_{n_i}} \in X$ be such that $\|A_{n_1 k_{n_1}} x_{k_{n_i}}\| = 1$. Suppose that $x_{k_{n_1}}, \ldots, x_{k_{n_{i-1}}}$ are choose in x and write

$$u_{i-1} = \sum_{j=1}^{i-1} A_{n_i, k_{n_j}} X_{k_{n_j}}.$$

Let $y_i \in X$ be such that $\|A_{n_i k_{n_i}} y_i\| = 1$. If we set $X_{k_{n_i}} = (i + \|u_{i-1}\|)\, y_i$, then

$$\|A_{n_i} x\| = \Big\| \sum_{j=1}^{i} A_{n_i k_{n_j}} x_{k_{n_j}} \Big\| = \|A_{n_i k_{n_i}} x_{k_{n_i}} + u_{i-1}\|$$

$$\geq \|A_{n_i k_{n_i}} x_{k_{n_i}}\| - \|u_{i-1}\| = i.$$

Thus $Ax \in l_\infty(X)$ and the proof is complete.

Proof of Theorem 2 – Let $A = (A_{nk}) \in (l_\infty(X), F)$. Define the matrix $A^r = (A^r_{n_k})$, $r \in N$, by

$$A^r_{nk} = \begin{cases} A_{nk}, & k \leq r, \\ \bar{\theta}, & k > r. \end{cases}$$

Thus each A^r is right-finite and considering, for each $z \in X$, the sequence whose k*th* element is z while others are θ, condition (2.1) of Theorem 1 holds. Hence $A^r \in (\omega(X), F)$ for each $r \in N$. Since $(l_\infty(X), F) \subset (l_\infty(X), c(Y))$. We have by Lemma 2 and hypothesis,

$$\|A^r - A\| \leq H \|A^r - A\|_1 = H \sup_n \|R_{n, r+1}\|$$
$$\to 0, \text{ as } r \to \infty.$$

In fact, the first term tends to 0 by (iii) of Lemma 2 and the second term tends to 0 by (iii) and (ii) of Lemma 2. This proves the first part of the theorem.

To prove the second part, let $F = l_\infty(X)$. The operator matrix $\mathcal{J}$ is in $(l_\infty(X), l_\infty(X))$. We show that $\mathcal{J}$ cannot be approximated by any sequence of matrices in $(\omega(X), l_\infty(X))$ with respect to $\|\cdot\|_1$.

Suppose, if possible, that $\{A^r\} = \{(A^r_{nk})\}$ is a sequence of elements of $(\omega(X), l_\infty(X))$ such that $\|A^r - \mathcal{J}\|_1 \to 0$ as $r \to \infty$. Since $\|A^r_{nn} - I\| \leq \|A^r - \mathcal{J}\|_1$, it follows that $A^r_{nn} \to I$ as $r \to \infty$, uniformly in n. Hence, there is $r_0 \in N$ such that $\|A^r_{nn}\| > 1/2$ for all $r > r_0$ and all $n \in N$. This is a contradiction since each A^r is right-finite by Theorem 1. The result follows.

Proof of Theorem 3 – Sufficiency : We suppose that $A = (A_{nk})$ is lower triangular, $A_{nk} \in B(X)$ for all $n, k \in N$ and the diagonal elements A_{nn} are invertible in $B(X)$. First, we show that there is a unique $X = (X_{nk})$ with $X_{nk} \in B(X)$ for all $n, k \in N$, such that $AX = \mathcal{J}$, or equivalently, the system of operator equations

$$A_{ni} X_{ik} = \delta_{nk} I, \; n, k \in N. \qquad \ldots(4.2)$$

where δ_{nk} are the kronecker numbers, has a unique solution $X = (X_{nk})$.

For $n = 1 = k$, (4.2) shows that $A_{11} X_{11} = I$, so that we must have $X_{11} = A_{11}^{-1} \in B(X)$.

For $n = 1$, $k > 1$, we have $A_{11} X_{1k} = \bar{\theta}$, so that $X_{1k} = \bar{\theta}$ for each $k > 1$.

To determine X_{nk} for $n > 1$ and all $k \in N$, we proceed inductively. Suppose that the first $n - 1$ rows of X are determined uniquely such that

$$X_{mk} \in B(X) \text{ for } 1 \leq m \leq n - 1, \; k \in N \qquad \ldots(4.3)$$

and

$$X_{mk} = \bar{\theta}, \text{ for } 1 \leq m \leq n - 1, \; k > m. \qquad \ldots(4.4)$$

By (4.2), $\displaystyle\sum_{i=1}^{n} A_{ni} X_{ik} = \delta_{nk} I$

$$\Rightarrow A_{nn} X_{nk} = \delta_{nk} I - \sum_{i=1}^{n-1} A_{ni} X_{ik}$$

$$\Rightarrow X_{nk} = A_{nn}^{-1}\left(\delta_{nk} I - \sum_{i=1}^{n-1} A_{ni} X_{ik}\right) \qquad \ldots(4.5)$$

which is unique for all $k \in N$. Also $X_{nk} \in B(X)$ for all $n \in N$ by (4.3) and our hypothesis. In particular, $X_{nn} = A_{nn}^{-1}$, and $X_{nk} = \bar{\theta}$ for all $k > n$ by (4.4) and (4.5). Thus the nth row of X is uniquely determined and the induction is completed, which proves our assertion. The fact that X is also a right inverse of A, is an easy consequence of the above.

The necessity of the conditions (i) and (ii) follow smoothly from the facts that $AK = \mathcal{J} = XA$ and $X_{nk} \in B(X)$ for all $n, k \in N$, by making use of the open mapping theorem and the sufficiency part. The proof of Theorem 3 is complete.

Corollary – Let $A = (A_{nk}) \in \Delta(X)$. Then $A \in G(X)$ if and only if (i) A_{nn} is invertible in $B(X)$ for each $n \in N$, and (ii) $c_A = c(X)$.

Proof of Theorey 4 – Sufficiency : By virtue of Theorem 2, we can suppose, without loss of generality, that $A = (A_{nk}) \in (\omega(X), c(X) \cap \Delta(X)$ and proceed to show that $A \in \partial\, G(X)$, under the given hypotheses.

By Theorem 1, there exists $k_0 \in N$ such that $A_{nk} = \bar{\theta}$ for all $k > k_0$ and all $n \in N$. The conclusion follows on constructing a sequence $(A^r) = ((A^r_{nk}))$ of elements of $G(X)$ such that $\|A - A^r\|_1 \to 0$, as $r \to \infty$. By hypothesis, we can choose, for each $n = 1, 2, 3, \ldots, k_0$, a sequence $(A^r_{nn})_{r \in N}$ of invertible elements of $B(X)$ such that

$$\|A^r_{nn} - A_{nn}\| \to 0, \text{ as } r \to \infty.$$

For any $r \in N$, define the matrix $A^r = (A^r_{nk})$ by

$$A^r_{nk} = \begin{cases} A_{nk}, & n \neq k \\ r^{-1} I, & n = k > k_0, \\ A^r_{nn}, & n = k \leq k_0. \end{cases}$$

Then $A^r \in G(X)$ by the corollary to Theorem 3 and

$$\|A - A^r\|_1 = \sup_{\substack{m,n \in N \\ x_k \in S(X)}} \left\| \sum_{k=1}^{m} (A_{nk} x_k - A^r_{nk} x_k) \right\|$$

$$= \sup_{\substack{n \in N \\ x_n \in S(X)}} \|A_{nn} x_n - A^r_{nn} x_n\|$$

$$\leq \max \{r^{-1}, \|A_{11} - A^r_{11}\|, \ldots, \|A_{k_0 k_0} - A^r_{k_0 k_0}\|\}$$

$$\to 0, \text{ as } r \to \infty.$$

This completes the proof of sufficiency.

Necessity : Suppose that $A = (A_{nk}) \in \partial\, G(X)$. We can choose matrices $A^r = (A^r_{nk}) \in G(X)$, $r \in N$, such that

$$\|A - A^r\|_1 = \sup_{\substack{m,\, n \in N \\ x_k \in S(X)}} \left\| \sum_{k=1}^{m} (A_{nk} - A^r_{nk})\, x_k \right\| \to 0, \text{ as } r \to \infty$$

which implies that for fixed $n, k \in N$,

$$\|A_{nk} - A^r_{nk}\| = \sup_{x \in S(X)} \|(A_{nk} - A^r_{nk})\, x\| \leq \|A - A^r\|_1 \to 0, \text{ as } r \to \infty.$$

In particular, $\|A_{nn} - A^r_{nn}\| \to 0$, as $r \to \infty$, for each fixed $n \in N$. Since A^r_{nn} is invertible in $B(X)$ for all $n, r \in N$, by the corollary to Theorem 3, the necessity of the condition follows. This completes the proof.

Proof of Theorem 5 – It follows by the Corollary to Theorem 3 and the hypothesis on A that

$$\bigcup_{i \in N} \sigma(A_{ii}) \cup \{0\} \subset \sigma(A).$$

Conversely, wer show that if $\lambda \neq 0$ or $\lambda \notin \sigma(A_{ii})$ for any $i \in N$, then $\lambda \notin \sigma(A)$, or equivalently, $\lambda\mathcal{J} - A \in G(X)$. It is enough to check the conditions (i) and (ii) of the corollary to Theorem 3 for the operator matrix $\lambda\mathcal{J} - A$. Condition (i) is an immediate consequence of the choice of λ. For (ii), we show that $x \in c(X)$ whenever $(A - \lambda\mathcal{J})\, x \in c(X)$.

By Theorem 2, we can choose $B \in (\omega(X), c(X)) \cap \Delta(X)$ such that $\|A - B\|_1 < |\lambda|$. This implies that $A - B - \lambda\mathcal{J} \in x\, G(X)$. If $(A - \lambda\mathcal{J})\, x = Y \in c(X)$, then $(A - B - \lambda\mathcal{J})\, x = y - Bx \in c(X)$, since $B \in (\omega(x), c(X))$. Hence $x \in c(X)$ and the proof is complete.

References

1. R. G. Cook, *Infinite Matrices and Sequence Spaces,* Dover Publications, Inc., 1950.
2. I. J. Maddox, *Infinite Matrices of Operators,* Lecture Notes in Mathematics, Springer-Verlag, 1980.
3. M. R. Paremeswaran, *Proc. Indian Acad. Sci.* **36** (1952).
4. M. S. Ramanujan, *Math. Annalen* **159** (1965), 365-73.
5. D. Rath, *Bull. Cal. Math. Soc.* **76** (1984), 280-83.
6. A. Robinson, *Proc. London Math. Soc.* (second series), **52** (1952), 132-60.
7. N. K. Sharma, *Proc. Am. Math. Soc.* **35**(2) (1972), 515-18.
8. A. Wilansky and K. Zeller, *Illinois J. Math.* **2** (1958), 378-85.
9. B. Yood, *Ann. Math* (2) **50** (1949), 486-503.