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LOCALLY NEARLY PARALINDELÖF SPACES

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We define in this paper, NPL-sets in a topological space X and study their properties. A subset A ($\subset X$) is an NPL-set in X if for every open cover $\mathcal{U}$ of A, there is a collection $\mathcal{V}$ of open subsets of X such that $\mathcal{V}$ refines $\mathcal{U}$, $\mathcal{V}$ is locally countable and $A \subset \cup\{\text{int cl } V \mid V \in \mathcal{V}\}$. Every pair of disjoint NPL-sets in X, when X is nearly paralindelöf, can be separated by open nbds with disjoint closures. A locally nearly paralindelöf[9] space is one each point of which has a nbd whose closure is an NPL-set in X. X is locally nearly paralindelöf if and only if for each NPL-set in X, say C, and a regular open nbd U of C there is an open set V such that $C \subset V \subset \text{cl } V \subset U$ and cl V is an NPL-set in X. Also if X is locally nearly paralindelöf, then X is almost regular.

A topological space is called a P-space if every G_δ-set is open. A topological space is called an HP-space if it is Hausdorff and a P-space. In our earlier papers[1,2] an extensive investigation of spaces called nearly paralindelöf spaces was carried out. A topological space (X, τ) is called nearly paralindelöf if every open cover $\mathcal{U}$ of X admits a collection $\mathcal{V}$ of open subsets of X such that $\mathcal{V}$ is locally countable, $\mathcal{V}$ refines $\mathcal{U}$ and $X = \cup\{\text{int cl } V \mid V \in \mathcal{V}\}$. In the papers cited above, the author established various properties of nearly paralindelöf spaces, studied the subsets of such spaces and also the behaviour of such spaces under various types of functions.

In this paper and a sequel to follow we study in detail the local theory of such spaces. In this paper we introduce the concept of an NPL-set in X and study the same and its relation to other subsets of X. We notice that every pair of disjoint NPL-sets in X, when X is nearly paralindelöf, can be separated by open sets with disjoint closures. We define the notion of locally nearly paralindelöf spaces and show such spaces are almost regular.

All spaces discussed in this paper are HP-spaces. The letter N stands for the set of natural numbers. The abbreviation 'nbd' stands for neighbourhood. When several topologies are involved, if the context needs it, we write σ-cl A (σ-int A) to emphasize that the closure (interior) of A is taken in the topology σ. Except where the contrary is stated, the topology is understood to be τ. If A is a subset of X and τ is a topology on X, then τ/A stands for the relative topology of τ on A. □ marks the end of proof.

An open subset U of X is 'regular open if' int cl $U = U$. The set of all regular open subsets of X forms a base for a topology, usually denoted by τ_s; τ_s is coarser

than τ and it is called the semi regularization of τ. The complement of a regular open subset of X is called regular closed. A space is called almost regular if, whenever B is a regular closed subset of X and $x \notin B$, then there are disjoint open subsets U and V such that $B \subset U$ and $x \in V$.

Definition 1 – Let (X, τ) be a topological space. Let $A \subset X$.

(i) A is called a 'nearly paralindelöf set' if $(A, \tau/A)$ is nearly paralindelöf.

(ii) A is called an 'NPL-set in X' if every open cover $\mathfrak{U}$ of A admits an open refinement $\mathfrak{V}$ (not necessarily a cover of A) which is locally countable and $A \subset \cup\{\text{int cl } V | V \in \mathfrak{V}\}$.

On some occasions we write NPL-set instead of NPL-set in X when the context is clear. It must be noted that in (ii) above, $\mathfrak{V}$ is locally countable in X, that is, each point of X has an open nbd which meets only countably many members of $\mathfrak{V}$.

Proposition 2 – In a space X, every NPL-set in X is closed.

Proof : Let A be an NPL-set in X and $y \notin A$. For each $x \in A$, there are disjoint open nbds U_x and W_x of x and y respectively. $\mathfrak{U} = \{U_x \mid x \in A\}$ is an open cover of A so that it admits a locally countable collection $\mathfrak{V}$ of open subsets of X such that $\mathfrak{V}$ refines $\mathfrak{U}$ and $A \subset \cup\{\text{int cl } V \mid V \in \mathfrak{V}\}$ ($= P$, say). Since $\mathfrak{V}$ is locally countable there is an open nbd M_y of y which meets only countably many members of $\mathfrak{V}_1 = \{\text{int cl } V \mid V \in \mathfrak{V}\}$. If these members are $\{\text{int cl } V_i \mid i \in N\}$, then each $V_i \subset U_{x(i)}$ for $x(i) \in A$, $i \in N$. Also $(\text{int cl } U_{x(i)}) \cap W_{x(i)}$ so that if $Q = M_y \cap (\cap\{W_{x(i)} \mid i \in N\})$, then P and Q are disjoint open nbds of A and y respectively.

Proposition 3 – If $\{A_i \mid i \in N\}$ is a countable family of subsets of X each of which is an NPL-set in X, then $\cup\{A_i \mid i \in N\}$ is an NPL-set in X.

Proposition 4 – A subset A ($\subset X$) is an NPL-set in X if and only if every regular open cover $\mathfrak{U}$ of A has a locally countable open refinement $\mathfrak{V}$ that covers A.

Proof : Let A be an NPL-set in X. Let $\mathfrak{U}$ be a regular open cover of A. Since $\mathfrak{U}$ is an open cover of A, there is a locally countable collection $\mathfrak{V}'$ of open subsets of X such that $\mathfrak{V}'$ refines $\mathfrak{U}$ and $A \subset \cup\{\text{int cl } V | V \in \mathfrak{V}'\}$. If $V \in \mathfrak{V}'$, then $V \subset U$ for some $U \in \mathfrak{U}$ so that int cl $V \subset$ int cl $U = U$. Hence $\mathfrak{V} = \{\text{int cl } V | V \in \mathfrak{V}\}$ is an open refinement of $\mathfrak{U}$ such that $\mathfrak{V}$ covers A and further $\mathfrak{V}$ is locally countable.

Conversely, let $\mathfrak{U}$ be an open cover of A. Then $\mathfrak{W} = \{\text{int cl } U | U \in \mathfrak{U}\}$ is a regular open cover of A so that there is a locally countable open refinement $\mathfrak{V}$. Consider $V \in \mathfrak{V}$; then there is $W(V) \in \mathfrak{W}$ and $U(V) \in \mathfrak{U}$ such that $V \subset W(V) =$ int cl $U(V)$. Let us write $G(V) = V - [\text{cl } U(V) - U(V)] = V \cap U(V)$ and $\mathfrak{G} = \{G(V) | V \in \mathfrak{V}\}$. Since $G(V) \subset V$, $\mathfrak{G}$ is locally countable. Also int cl $U(V) =$ int cl V. Thus $A \subset \cup\{V | V \in \mathfrak{V}\} \subset \cup\{\text{int cl } V | V \in \mathfrak{V}\} = \cup\{\text{int cl } G(V) | G(V) \in \mathfrak{G}\}$. Thus $\mathfrak{G}$ is a locally countable collection of open sets that refines $\mathfrak{U}$ and $A \subset \cup\{\text{int cl } G | G \in \mathfrak{G}\}$.

Corollary 5 – A subset A ($\subset X$) is an NPL-set in X if and only if every regular open cover $\mathfrak{U}$ of A has a locally countable regular open refinement $\mathfrak{V}$ that covers A.

Theorem 6 – Let X be an almost regular space. Let $A \subset X$. Then the following are equivalent :

(i) A is an NPL-set in X.

(ii) Every open cover $\mathfrak{U}$ of A in X admits a collection $\mathfrak{V}$ of open subsets of X such that $\mathfrak{V}$ refines $\mathfrak{U}$, $\mathfrak{V}$ is locally countable and $A \subset \cup\{\text{cl } V | V \in \mathfrak{V}\}$.

(iii) Same as (ii) except that $\mathfrak{U}$ is required to be a regular open cover.

Proof : (i) $\Rightarrow$ (ii). This implication is true even without X being almost regular.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Let $\mathfrak{U} = \{U_\alpha | \alpha \in \Delta\}$ be a regular open cover of A. For each $x \in A$, there is an $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Then there is a regular open set V_x such that $x \in V_x \subset \text{cl } V_x \subset U_{\alpha(x)}$. Thus we get a regular open cover of A. $\mathfrak{V}_1 = \{V_x | x \in A\}$. By hypothesis, there is a collection $\mathfrak{V}$ of open subsets of X such that $\mathfrak{V}$ is locally countable $\mathfrak{V}$ refines $\mathfrak{V}_1$ and $A \subset \cup\{\text{cl } W | W \in \mathfrak{V}\}$. In this way the proof parallels that of Theorem 1.4 of our earlier paper[1].

In a topological space X, a subset A ($\subset X$) is called an APL-set in X if for every open cover $\mathfrak{U}$ of A, there is an open collection $\mathfrak{V}$ of subsets of X such that $\mathfrak{V}$ refines $\mathfrak{U}$, $\mathfrak{V}$ is locally countable and $A \subset \cup\{\text{cl } V | V \in \mathfrak{V}\}$. This gives

Proposition 7 – Let A be a subset of X. (i) If A is an NPL-set in X then it is an APL-set in X. (ii) The converse holds if X is almost regular.

Proposition 8 – Let A be a subset of X. (i) If A is an NPL-set in X, then $(A, \tau_s/A)$ is paralindelöf. (ii) The converse holds if X is almost regular and A is regular closed.

Proof : (i) Direct consequence of Corollary 5.

(ii) Let $\mathfrak{U}$ be a regular open cover of A where $A = \text{cl } F$ for some open subset $F \subset X$. Then $\mathfrak{U}$ is a τ_s-open cover of A. $\{U \cap A | U \in \mathfrak{U}\}$ is a (τ_s/A)-open cover of A. It has a (τ_s/A)-open refinement $\mathfrak{U}_A = \{V(A, \alpha) | \alpha \in \Delta\}$ such that $\mathfrak{V}_A$ is locally countable (w.r.t. τ_s/A) in A. For each $\alpha \in \Delta$, there is a $V_\alpha \in \tau_s$, such that $V(A, \alpha) = V_\alpha \cap A$. Form $\mathfrak{V}_F = \{V_\alpha \cap F | \alpha \in \Delta\}$. $\mathfrak{V}_F$ refines $\mathfrak{U}$. V_F is an open collection of subsets of X. Since $\cup\{V(A, \alpha) | \alpha \in \Delta\} = \cup\{V_\alpha \cap A \mid \alpha \in \Delta\} = A$, $A \subset \cup\{\sigma\text{-cl}(V_\alpha \cap A) | \alpha \in \Delta\}$ where $\sigma = \tau/A$. Also $\sigma\text{-cl}(V_\alpha \cap A) = \text{cl}(V_\alpha \cap A) \cap A = \text{cl}(V_\alpha \cap \text{cl } F) \cap \text{cl } F = \text{cl}(V_\alpha \cap F) \cap \text{cl } F = \text{cl}(V_\alpha \cap F)$. Thus $A \subset \cup\{\text{cl}(V_\alpha \cap F) | \alpha \in \Delta\} = \cup\{\text{cl } W | W \in \mathfrak{V}_F\}$. Since each $V_\alpha \cap F \subset V_\alpha \cap A$, $\mathfrak{V}_F$ is locally countable (w.r.t. τ_s/A) in A so that each point $x \subset A$ has an open abd M_x that meets only countably many members of $\mathfrak{V}_F$. It should be noted that $X - A$ is an open set that meets no member of $\mathfrak{V}_F$. Hence by Theorem 6(iii), the result follows.

Proposition 9 – Let X be a nearly paralindelöf space. If A is a regular closed subset of X, then A is an NPL-set in X.

Proof : Let $A = \text{cl } F$ where F is open. Let $\mathfrak{U}$ be an open cover of A. Then $\mathfrak{U}' = \mathfrak{U} \cup \{X - A\}$ is an open cover of X. $\mathfrak{U}'$ has an open refinement $\mathfrak{V}'$ (not necessarily a cover of X) such that $X = \cup\{\text{cl } V | V \in \mathfrak{V}'\}$ and $\mathfrak{V}'$ is locally countable.

Let $\mathcal{V}_1$ be all those sets in $\mathcal{V}$ which are contained in $X - \text{cl}(F)$. Then $\cup\{\text{cl } V | V \in \mathcal{V}_1\} = \text{cl}(\cup\{V | V \in \mathcal{V}_1\} \subset X - \text{int cl } F$ so that $\cup\{\text{cl } V | V \in \mathcal{V} - \mathcal{V}_1\} \supset \text{cl } F$. Since X is almost regular (Theorem 1.3 of Daniel Thanapalan[1]), an application of Theorem 6 gives the result.

Proposition 10 – Let $B \subset A \subset X$ and A an open subset of X. If B is an NPL-set in X, then B is an NPL-set in A.

Proof : Let τ be the topology on X and $\tau/A = \sigma$. Let $\mathcal{U}$ be a σ-open cover of B. Since A is open, $\mathcal{U}$ is an open cover of B. Then $\mathcal{U}$ has an open refinement $\mathcal{V}$ (not necessarily a cover) which is locally countable and $B \subset \cup\{\text{int cl } V | V \in \mathcal{V}\}$. If $V \in \mathcal{V}$, there is an $U \in \mathcal{U}$ such that $V \subset U$ (notice that $U \subset A$). Thus each element of $\mathcal{V}$ is σ-open and $B = B \cap A \subset \cup\{(\text{int cl } V) \cap A | V \in \mathcal{V}\} = \cup\{\sigma\text{-int } \sigma\text{-cl})(V \cap A) | V \in \mathcal{V}\} = \cup\{\sigma\text{-int } \sigma\text{-cl } V | V \in \mathcal{V}\}$, and hence the result

Any regular closed subset as a subspace of an almost regular space is almost regular. For want of an immediate reference, let us indicate the proof.

Proposition 11 – If (X, τ) is almost regular and A is a regular closed subset of X, then $(A, \tau/A)$ is almost regular.

Proof : Let B ($\subset A$) be regular closed in $(A, \tau/A)$ and $x \notin B$, but in A. Then there exists a set $V \in \tau/A$ such that $B = \sigma\text{-cl } V$ where $\sigma = \tau/A$. There is a set $U \in \tau$ such that $V = U \cap A$ so that $\sigma\text{-cl } V = \text{cl}(U \cap F)$. Thus B is regular closed in (X, τ). Hence there are disjoint open sets P and Q such that $x \in P$, $B \subset Q$. Setting $P_1 = P \cap A$, $Q_1 = Q \cap A$, we get $P_1 \in \sigma$, $Q_1 \in \sigma$, $x \in P_1$, $B \subset Q_1$ and $P_1 \cap Q_1 = \phi$.

Proposition 12 – Let (X, τ) be an almost regular space. Let A be a regular closed subset of X. Then the following are equivalent :

(i) $(A, \tau/A)$ is nearly paralindelöf.

(ii) A is an NPL-set in X.

Proof : (i) $\Rightarrow$ (ii). $(A, \tau/A)$ is nearly paralindelöf so that $(A, \tau/A)$ is almost paralindelöf and then it can be easily verified that A is an APL-set in X. Since X is almost regular, (ii) follows as a consequence of Proposition 7.

(ii) $\Rightarrow$ (i). If A is an NPL-set in X, A is an APL-set in X, so that $(A, \tau/A)$ is almost paralindelöf. By Proposition 11, $(A, \tau/A)$ is almost regular and hence $(A, \tau/A)$ is nearly paralindelöf. □

Proposition 13 – The following are equivalent for a space (X, τ) :

(i) X is nearly paralindelöf.

(ii) The closure of every open subset of X is an NPL-set in X.

(iii) The closure of every open subset of X is nearly paralindelöf.

Proof : That (i) $\Rightarrow$ (ii) follows from Proposition 9. That (ii) $\Rightarrow$ (i) follows obviously. Suppose (ii) is true; then (i) holds so that the space is almost regular. Thus on applying Proposition 12, (iii) follows. Similarly (iii) $\Rightarrow$ (ii).

Proposition 14 – If B is regular closed in X, $B \subset A$ and A is an NPL-set in X, then B is an NPL-set in X.

PROOF : The proof of this result is similar to Proposition 9.

Proposition 15 – If A is an NPL-set in X, B is regular open in X and $B \subset A$, then $A - B$ is an NPL-set in X.

PROOF : The proof of this result is similar to Proposition 9.

Proposition 16 – Let B be an NPL-set in X. For any $x \in B$, and any regular open set A such that $x \in A \subset B$, there is an open set W such that $x \in W \subset \text{cl } W \subset A$.

PROOF : For each $y \in B - A$, there are open nbds W_y and U_y of x and y respectively such that $U_y \cap W_y = \phi$. Further we can assume each $W_y \subset A$. The collection $\mathfrak{U} = \{U_y \, y \in B - A\}$ is an open cover of $B - A$. Hence there exists a collection $\mathfrak{V}$ of open subsets of X such that $\mathfrak{V}$ is locally countable, $\mathfrak{V}$ refines $\mathfrak{U}$ and $B - A \subset \cup\{\text{int cl } V | V \in \mathfrak{V}\}$. Since $\mathfrak{V}$ is locally countable, there is an open nbd M_x of x such that M_x meets only countably many members of $\mathfrak{V}$, say, V_i $(i \in N)$. Let each $V_i \subset U_{y(i)}$ for a suitable $y(i) \in B - A (i \in N)$. Also $W_{y(i)} \cap \text{int cl } U_{y(i)} = \phi$ so that if $w = M_x \cap (\cap \{W_{y(i)} \mid i \in N\})$, then W meets none of the collection $\{\text{int cl } V | V \in \mathfrak{V}\}$. Thus we get $x \in W \subset \text{cl } W \subset A$.

Proposition 17 – Let (X, τ) be a nearly paralindelöf space, A an NPL-set in X and $p \notin A$. Then there are nbds of p and A with disjoint closures.

PROOF : Let $y \in A$. There are disjoint regular open nbds U_u and W_y of y and p respectively. Let $\mathfrak{U} = \{U_y | y \in A\}$. Since $\mathfrak{U}$ is a regular open cover of A, by Proposition 4, there is a locally countable open refinement $\mathfrak{V}$ of $\mathfrak{U}$. Now there is an open nbd M of p such that M meets only countably many members of $\mathfrak{V}$, say, V_i $(i \in N)$. There are $y(i) \in A$ such that $V_i \subset U_{y(i)}$, $(i \in N)$; now $W_{y(i)} \cap U_{y(i)} = \phi$ so that $W = M \cap (\cap\{W_{y(i)} \mid i \in N\})$ is open nbd of p such that $W \cap V = \phi$ for each $V \in \mathfrak{V}$. Thus, $W \cup \{V | V \in v\} = \phi$. If $\cup\{V | V \in \mathfrak{V}\} = V'$, then $W \cap V' = \phi$ so that $A \subset V' \subset \text{cl } V'$ and $\text{cl } V' \cap W = \phi$. Since cl V' is a regular-closed subset of the nearly paralindelöf space X, cl V' is an NPL-set in X and $p \notin \text{cl } V'$. Repeating the above arguments for p and cl V' we get open sets N_1 and N_2 such that $\text{cl } V' \subset N_1$ $p \in N_2$ and $N_1 \cap N_2 = \phi$. Then $\text{cl } V' \cap \text{cl } N_2 = \phi$ where $A \subset V'$ and $p \in N_2$.

Theorem 18 – Let (X, τ) be nearly paralindelöf. Let P and Q be a pair of disjoint NPL-sets in X. Then there are nbds of P and Q with disjoint closures.

PROOF : Let $q \in Q$. Then, by Proposition 17, there are regular open nbds U_q and W_q of q and P respectively with disjoint closures. Thus $\mathfrak{U} = \{U_q | q \in Q\}$ is a regular open cover of Q which is an NPL-set in X so that there is a locally countable open refinement $\mathfrak{V}$ of $\mathfrak{U}$ by Proposition 4. Indeed $Q \subset \cup\{V | V \in \mathfrak{V}\}$ and for each $V \in \mathfrak{V}$, cl $V \subset$ cl U_q for a suitable $q \in Q$. Since $\text{cl } W_q \cap \text{cl } U_q = \phi$, $\text{cl } V \cap \text{cl } W_q = \phi$. If we take $R = \cup\{V | V \in \mathfrak{V}\}$ $R \supset Q$ and $\text{cl } R \cap P = \phi$. Since X is nearly paralindelöf, cl R is an NPL-set in X, so that, by a similar argument, there is an open set S such that $\text{cl } R \subset S$ and $\text{cl } S \cap P = \phi$. Thus $Q \subset R \subset \text{cl } R \subset S \subset \text{int cl } S \subset X - P$ so that if $X - \text{cl } S = T$, then $Q \subset R$, $P \subset T$ and $\text{cl } R \cap \text{cl } T = \phi$.

Corollary 19 – Let X be a nearly paralindelöf space. Then every pair of disjoint regular-closed subsets of X can be separated by open sets with disjoint closures.

PROOF : If A and B are disjoint regular-closed subsets of X, then they are disjoint NPL-sets in X so that Theorem 18 applies.

Theorem 20 – Let (X, τ) be a nearly paralindelöf space and A and B disjoint NPL-sets in X. Then

(i) there is a real-valued function f continuous on x which is identically 0 on A and 1 on B;

(ii) (X, τ) is completely Hausdorff.

PROOF : (i) This is similar to the proof of Urysohn's Lemma for normal spaces.

(ii) Follows from (i).

Definition 21 – A space (X, τ) is locally nearly paralindelöf if each point of X has a nbd whose closure in X is an NPL-set in X.

In the above definition, the nbd need not necessarily be open; further we could as well have taken " .... if each point of X has a nbd which is an NPL-set in X".

Recall that a subset $A (\subset X)$ is C_1-closed if every open cover of A has a countable subfamily $\mathcal{V}(\subset \mathcal{U})$ such that $A \subset \cup \{\text{int cl } V | V \in \mathcal{V}\}$. A subset A $(\subset X)$ is HP-closed relative to X if every open cover of A has a countable subfamily $\mathcal{V}$ $(\subset \mathcal{U})$ such that $A \subset \cup \{\text{cl } V | V \in \mathcal{V}\}$. The definitions of C_1-closed sets and sets HP-closed relative to X occur in Daniel Thanapalan and Raghavan[3]. In general every C_1-closed set is HP-closed relative to X. On the other hand, if X is almost regular, every subset HP-closed relative to X is C_1-closed. If A is C_1-closed then A is an NPL-set in X. If A is an NPL-set in X, then A is an APL-set in X. There exists an almost paralindelöf space which is not nearly paralindelöf and therefore there exists an APL-set (in X) which is not a NPL-set in X. There exists a nearly paralindelöf space which is not HP-closed and hence there exists an NPL-set in X which is not HP-closed relative to X (and hence not C_1-closed).

Remark 22 : It can be easily verified that if B has a nbd U such that cl U is an NPL-set in X, then B has a regular open nbd V such that cl V is an NPL-set in X and $U \subset V \subset$ cl U.

Proposition 23 – Let X be almost regular. Let $\{F_\alpha | \alpha \in \Delta\}$ be a locally countable collection of open subsets of X such that cl F_α is an NPL-set in X for each $\alpha \in \Delta$. Then $\cup \{\text{cl } F_\alpha | \alpha \in \Delta\}$ is an NPL-set in X.

PROOF : Let $\mathcal{U}$ be a regular open cover of $H = \cup \{\text{cl } F_\alpha | \alpha \in \Delta\}$. Then $\mathcal{U}$ is a regular open cover of cl F_α for each $\alpha \in \Delta$. By Proposition 4 $\mathcal{U}$ has a locally countable refinement $\mathcal{V}_\alpha$ $(\alpha \in \Delta)$. Form $\mathcal{W}_\alpha = \{F_\alpha \cap V | V \in \mathcal{V}_\alpha\}$. $\mathcal{W}_\alpha$ is locally countable and an open collection such that if $W \in \mathcal{W}_\alpha$, then $W = F_\alpha \cap V$ for some $V \in \mathcal{V}_\alpha$ and $W = F_\alpha \cap V | V \subset U$ for some $U \in \mathcal{U}$. Thus $\mathcal{W}_\alpha$ refines $\mathcal{U}$. Also $\cup \{\text{cl } (F_\alpha \cap V | V \in \mathcal{V}_\alpha\} = \text{cl } (\cup \{F_\alpha \cap V \in \mathcal{V}_\alpha\}) = \text{cl } (F_\alpha \cap (\cup \{V | V \in \mathcal{V}_\alpha\})) =$ cl F_α. Let us take $\mathcal{W} = \cup \{\mathcal{W}_\alpha | \alpha \in \Delta\}$. Then, if $x \in X$, there is an open nbd G_x of x such that G_x meets cl F_β for each $\beta \in \Delta(x)$ where $\Delta(x)$ is a suitable countable

subset of Δ and $G_x \cap \text{cl } F_\alpha = \phi$ for each $\alpha \in \Delta - \Delta(x)$. Since $\mathcal{V}_\beta$ is locally countable, there exists an open nbd of x, say, $P(x, \beta)$, which meets only countably many members of $\mathcal{V}_\beta$. Notice that $\beta \in \Delta(x)$. Set $M_x = G_x \cap (\cap \{P(x, \beta) | \beta \in \Delta(x)\})$. Thus M_x meets only countably many members of $\mathcal{W}$, so that $\mathcal{W}$ is locally countable. Also $\cup\{\text{cl } W | W \in \mathcal{W}\} = \cup\{\text{cl } F_\alpha | \alpha \in \Delta\} = H$. Further $\mathcal{W}$ refines $\mathcal{U}$. Now Theorem 6(iii) applies.

Proposition 24 – The following are equivalent for a space (X, τ):

(i) The space X is locally nearly paralindelöf;

(ii) For each $x \in X$ and each nbd U of x there is an open nbd V (of x) whose closure is an NPL-set in X such that $x \in V \subset \text{cl } V \subset \text{int cl } U$.

(iii) Same as (ii) except that U is required to be regular open.

Proof : (i) $\Rightarrow$ (ii). Since (X, τ) is locally nearly paralindelöf, for each $x \in X$, there is an open nbd W of x such that cl W is an NPL-set in X. The set int cl $(W \cap U)$ is regular open and $\subset$ cl W so that by Proposition 16 there is an open set V such that $x \in V \subset \text{cl } V \subset \text{int cl } (W \cap U) \subset \text{int cl } U$.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Obvious.

Corollary 25 – Let X be a locally nearly paralindelöf space. Then X is almost regular.

Proposition 26 – The following are equivalent for a space (X, τ) :

(i) X is locally nearly paralindelöf;

(ii) For each NPL-set in X, say C, and each regular open nbd U of C, there is an open nbd V such that $C \subset V \subset \text{cl } V \subset U$ and cl V is an NPL-set in X.

Proof : (i) $\Rightarrow$ (ii). By Proposition 25, the space is almost regular and hence for each $x \in C$, we can find an open set V_x such that $x \in V_x \subset \text{cl } V_x \subset U$ and that cl V_x is an NPL-set in X, by Proposition 24(iii). Since C is an NPL-set in X, there is a family $\mathcal{W}$ of open subsets of X such that $\mathcal{W}$ refines $\mathcal{V} = \{V_x | x \in C\}$. $\mathcal{W}$ is locally countable and $C \subset \cup\{\text{int cl } W | W \in \mathcal{W}\}$. For each W, cl $W \subset$ cl V_x for some $x \in C$. Thus cl W is an NPL-set in X for each $W \in \mathcal{W}$. Hence $\cup\{\text{cl } W | W \in \mathcal{W}\}$ is an NPL-set in X by Proposition 23. If we write $\cup\{\text{int cl } W | W \in \mathcal{W}\} = V$, then cl V is an NPL-set in X so that $C \subset V \subset \text{cl } V \subset U$.

(ii) $\Rightarrow$ (i). Take the set C to be a singleton, then the result follows.

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