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ON GENERALIZED ITÔ TYPE STOCHASTIC FUNCTIONAL INTEGRAL EQUATIONS IN TWO INDEPENDENT VARIABLES

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We investigate the existence and uniqueness of solutions of a general form of Itô type stochastic functional integral equations in two independent variables. The equations considered in this paper are of more general type and in the special cases include the equations studied by J. Turo and by the present authors.

1. INTRODUCTION

In this paper we are concerned with the study of the nonlinear Itô type stochastic functional integral equation in two independent variables of the form

$$z(x,y,\omega) = F(x,y,A_1z,A_2z,A_3z,B_1z,B_2z,B_3z,Cz,\omega) = (Tz)(x,y,\omega) \quad \ldots(1.1)$$

where

$$A_1z = \int_0^{a(x,y)} \int_0^{b(x,y)} f_1(x,y,s,t,z(s,t,\omega),\omega)\, dsdt$$

$$A_2z = \int_0^{c(x,y)} f_2(x,y,s,z(s,q(x,y),\omega),\omega)\, ds$$

$$A_3z = \int_0^{d(x,y)} f_3(x,y,t,z(p(x,y),t,\omega),\omega)\, dt$$

$$B_1z = \int_0^{a(x,y)} \int_0^{b(x,y)} g_1(x,y,s,t,z(s,t,\omega),\omega)\, d\beta\ (s,t,\omega)$$

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$$B_2 z = \int_0^{c(x,y)} g_2(x,y,s,z(s,q(x,y),\omega),\omega)\, d\beta\,(s,y,\omega)$$

$$B_3 z = \int_0^{d(x,y)} g_3(x,y,t,z(p(x,y),t,\omega),\omega)\, d\beta\,(x,t,\omega)$$

$$C = z(g(x,y),\, h(x,y),\, \omega).$$

Here $z(x,y,\omega)$ is the unknown random function on

$D \times \Omega$, for $(x,y) \in D = \{(x,y) \mid x \in [0,X] = I_X,\ y \in [0,Y] = I_Y\}$ and $\omega \in \Omega$ the supporting set of the complete probability measure space $(\Omega, \mathcal{A}, P)$, into $R = (-\infty, \infty)$; $\beta(x,y,\omega)$ is a Wiener process in two independent variables defined for $(x,y) \in D$; f_i, g_i, $(i = 1, 2, 3)$ and F are given random functions defined on $\Delta_i\ X\ R\ X\ \Omega$ into R, $i = 1, 2, 3$ and $D\ X\ R^7\ X\ \Omega$ into R respectively where

$$\Delta_1 = \{(x,y,s,t) : 0 \le s \le x < \infty,\ 0 \le t \le y < \infty\}$$

$$\Delta_2 = \{(x,y,s) : 0 \le s \le x < \infty,\ 0 \le y < \infty\}$$

and

$$\Delta_3 = \{(x,y,t) : 0 \le x < \infty,\ 0 \le t \le y < \infty\}$$

and a, b, c, d, g, h, p and q are scalar functions defined on D into $R_+ = [0, \infty)$.

The special case of (1.1) when $f_2 = f_3 = g_2 = g_3 = 0$ is studied by J. Turo[13] by using the general method of successive approximations first introduced by T. Wazewski in[15]. In a recent paper[4] the present authors have studied equation (1.1) when $f_2 = f_3 = g_2 = g_3 = 0$ and F depends on some parameter. The equation (1.1) is of more general type and it contains as a special case the well known Itô type stochastic differential equation in two independent variables studied by many authors in the literature, see[10-12,16]. The general formulation of (1.1) is in fact motivated by recent results obtained by various authors on generalised version of Itô type stochastic integral equations in[2,4-7,13], see also[8,9,14]. The aim of the present paper is to obtain conditions which guarantee the existence, uniqueness and continuous dependence of the solution on the right hand side of equation (1.1). The main tool employed in our analysis is based on the general method of successive approximations introduced by T. Wazewski[15] and recently used by J. Turo[15] to study the special form of (1.1).

2. Preliminaries and Statement of Results

Let $\mathcal{A}_{xy}$, $(x,y) \in D$ be a σ-algebra of subsets of Ω such that $\mathcal{A}$ at $\subset \mathcal{A}_{xy}$ for $c \le s \le x \le X$, $0 \le t \le y \le Y$ for every $(x,y) \in D$, $\beta(x,y,\omega)$ is $\mathcal{A}_{xy}$-measurable and for every $(x,y) \in D$ the Wiener measure of the rectangle $[x,x_1]\ X\ [y,y_2] \subset D$, $x_1 = x + \lambda_1$, $y_2 = y + \lambda_2$ which can be defined by

$$\beta(x_1,y_2,\omega) - \beta(x_1,y,\omega) - \beta(x,y_2,\omega) + \beta(x,y,\omega)$$

is independent of $\mathcal{A}_{x_1 y}$ and $\mathcal{A}_{xy_2}$ for $\lambda_1, \lambda_2 \geq 0$.

Let $C(D,L_2)$ denote the space of all continuous maps $z : D \to L_2(\Omega, \mathcal{A}_{xy}, P)$. A random process z is called a solution of the stochastic functional integral equation (1.1) if $z \in C(D,L_2)$ and satisfies the equation (1.1) P - *a.e.*

We first list the following assumptions used in our subsequent discussion.

(H_1) Let,

(i) $f_1(x,y,s,t,u,.)$, $g_1(x,y,s,t,u,.)$ are $\mathcal{A}_{st}$-measurable for each $(x,y,s,t) \in \Delta_1$, $u \in C(D,L_2)$ and are continuous as maps from Δ_1 into $L_2(\Omega, \mathcal{A}, P)$;

(ii) $f_2(x,y,s,u,.)$, $g_2(x,y,s,u,.)$ are $\mathcal{A}_{sy}$-measurable for each $(x,y,s) \in \Delta_2$, $u \in C(D,L_2)$ and are continuous as maps from Δ_2 into $L_2(\Omega, \mathcal{A}, P)$;

(iii) $f_3(x,y,t,u,.)$, $g_3(x,y,t,u,.)$ are $\mathcal{A}_{xt}$-measurable for each $(x,y,t) \in \Delta_3$, $u \in C(D,L_2)$ and are continuous as maps from Δ_3 into $L_2(\Omega, \mathcal{A}, P)$;

(iv) $F(x,y,u_1,u_2,u_3,u_4,u_5,u_6,u_7,.)$ be $\mathcal{A}_{xy}$-measurable for each $(x,y) \in D$, $u_j \in C(D,L_2)$ and is continuous in x and y uniformly in u_j, $j = 1, 2, \ldots, 7$;

(v) for $(x,y) \in D$, $a(x,y)$, $c(x,y)$, $p(x,y)$, $g(x,y) \in I_x$ and $b(x,y)$, $d(x,y)$, $q(x,y)$, $h(x,y) \in I_y$.

(H_2) Let,

1^0 there exist functions k_j, m_j, $l \in C(D,R_+)$, $j = 1, 2, \ldots, 6$ such that

$$|f_1(x,y,s,t,u,\omega) - f_1(x,y,s,t,\bar{u},\omega)| \leq m_1(x,y)|u - \bar{u}|, \text{ for } (x,y,s,t) \in \Delta_1$$

$$|f_2(x,y,s,u,\omega) - f_2(x,y,s,\bar{u},\omega)| \leq m_2(x,y)|u - \bar{u}|, \text{ for } (x,y,s) \in \Delta_2$$

$$|f_3(x,y,t,u,\omega) - f_3(x,y,t,\bar{u},\omega)| \leq m_3(x,y)|u - \bar{u}|, \text{ for } (x,y,t) \in \Delta_3$$

$$|g_1(x,y,s,t,u,\omega) - g_1(x,y,s,t,\bar{u},\omega)| \leq m_4(x,y)|u - \bar{u}|, \text{ for } (x,y,s,t) \in \Delta_1$$

$$|g_2(x,y,s,u,\omega) - g_2(x,y,s,\bar{u},\omega)| \leq m_5(x,y)|u - \bar{u}|, \text{ for } (x,y,s) \in \Delta_2$$

$$|g_3(x,y,t,u,\omega) - g_3(x,y,t,\bar{u},\omega)| \leq m_6(x,y)|u - \bar{u}|, \text{ for } (x,y,t) \in \Delta_3$$

$$|F(x,y,u_1,u_2,u_3,u_4,u_5,u_6,u_7,\omega) - F(x,y,\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5,\bar{u}_6,\bar{u}_7,\omega)|$$

$$\leq \sum_{j=1}^{6} k_j(x,y)|u_j - \bar{u}_j| + l(x,y)|u_7 - \bar{u}_7|$$

for $(x,y) \in D$ and u_j, $\bar{u}_j$, u, $\bar{u} \in C(D,L_2)$, $j = 1, 2, \ldots, 7$,

2^0 $f_1(x,y,s,t,0,.)$, $g_1(x,y,s,t,0,.) \in L_2(\Omega, \mathcal{A}_{st}, P)$, for $(x,y,s,t) \in \Delta_1$

$f_2(x,y,s,0,.)$, $g_2(x,y,s,0,.) \in L_2(\Omega, \mathcal{A}_{sy}, P)$, for $(x,y,s) \in \Delta_2$

$f_3(x,y,t,0,.)$, $g_3(x,y,t,0,.) \in L_2(\Omega, \mathcal{A}_{xt}, P)$, for $(x,y,t) \in \Delta_3$

$F(x,y,0,0,0,0,0,0,0,.) \in L_2(\Omega, \mathcal{A}_{xy}, P)$, for $(x,y) \in D$.

We shall use the following linear operators in the proofs of our results :

$$(K_1u)\ (x,y) \overset{df}{=} k_1(x,y) \int_0^{a(x,y)} \int_0^{b(x,y)} u(s,t)\ dsdt,\ (x,y) \in D,$$

$$(K_2u)\ (x,y) \overset{df}{=} k_2(x,y) \int_0^{c(x,y)} u(s,q(x,y))\ ds,\ (x,y) \in D,$$

$$(K_3u)\ (x,y) \overset{df}{=} k_3(x,y) \int_0^{d(x,y)} u(p(x,y),t)\ dt,\ (x,y) \in D,$$

$$(Lu)\ (x,y) \overset{df}{=} l(x,y)\ u(g(x,y),h(x,y)),\ (x,y) \in D.$$

We put

$$k_1(x,y) = 14\,[\bar{k}_1^2\,(x,y)\ \bar{m}_1^2\,(x,y)\ a(x,y)\ b(x,y) + \bar{k}_4^2\,(x,y)\ \bar{m}_4^2\,(x,y)]$$

$$k_2(x,y) = 14\,[\bar{k}_2^2\,(x,y)\ \bar{m}_2^2\,(x,y)\ c(x,y) + \bar{k}_5^2\,(x,y)\ \bar{m}_5^2\,(x,y)]$$

$$k_3(x,y) = 14\,[\bar{k}_3^2\,(x,y)\ \bar{m}_3^2\,(x,y)\ d(x,y) + \bar{k}_6^2\,(x,y)\ \bar{m}_6^2\,(x,y)]$$

$$l(x,y) = 14\ \bar{l}^2\,(x,y). \qquad \ldots(2.1)$$

Suppose $C(D,R_+)$ denotes the class of all continuous and nonnegative scalar functions on D. We define

$$Su = \sum_{n=0}^{\infty} L^n u$$

with the pointwise convergence of the series in D, where $L^n = LL^{n-1}$, $n = 1, 2, \ldots$ and $L^0 = I$ the identity operator in $C(D, R_+)$. It follows, from the definition of the operator L, that

$$(L^n u)\ (x,y) = l_n\ (x,y)\ u(g_n(x,y),\ h_n(x,y)),\ n = 0,1,2,\ldots,$$

where

$$g_0(x,y) \overset{df}{=} x,\quad g_{n+1}\ (x,y) \overset{df}{=} g(g_n(x,y),\ h_n(x,y)),\ n = 0,1,2,\ldots,$$

$$h_0(x,y) \overset{df}{=} y,\quad h_{n+1}\ (x,y) \overset{df}{=} h(g_n(x,y),\ h_n(x,y)),\ n = 0,1,2,\ldots,$$

$$l_0(x,y) \overset{df}{=} 1,\ l_{n+1}\ (x,y) \overset{df}{=} \prod_{k=0}^{n} l(g_k(x,y),\ h_k(x,y)),\ n=0,1,2,\ldots,$$

for $(x,y) \in D$.

Now we state the following basic lemmas needed in the proof of our main results. Proofs of these lemmas are similar to those corresponding similar lemmas in[3,14]. We do not discuss the details.

Lemma 1 – Let

(i) $k_i, l, a, b, c, d, g, h, p, q, r \in C(D, R_+)$, $i = 1, 2, 3$ and $a(x,y)$, $c(x,y)$, $g(x,y)$, $p(x,y) \in I_x$, $b(x,y)$, $d(x,y)$, $h(x,y)$, $q(x,y) \in I_y$, for $(x,y) \in D$;

(ii) $s = Sr < \infty$, $\hat{s}_i = S\hat{k}_i < \infty$ where $\hat{k}_1(x,y) = k_1(x,y)\, a(x,y)\, b(x,y)$, $\hat{k}_2(x,y) = k_2(x,y)\, c(x,y)$, $\hat{k}_3(x,y) = k_3(x,y)\, d(x,y)$;

(iii) $s, \hat{s}_i \in C(D, R_+)$, $i = 1, 2, 3$ and there exist $\lambda_i \in R_+$ $i = 1, 2, 3$ such that $\hat{s}_1(x,y) \leq \lambda_1 xy$, $\hat{s}_2(x,y) \leq \lambda_2 x$, $\hat{s}_3(x,y) \leq \lambda_3 y$, $(x,y) \in D$.

Then

(1) there exists a unique solution $u_0 \in C(D, R_+)$ of the equation

$$u = SK_1u + SK_2u + SK_3u + Sr$$

in the class $M(D, R_+)$ of bounded and measurable functions defined on D,

(2) the function u_0 is the unique solution of the equation

$$u = K_1u + K_2u + K_3u + Lu + r$$

in the class

$$M(D, R_+, u_0) \stackrel{df}{=} \{u : u \in M(D, R_+), \|u\|_0 < \infty\}$$

where

$$\|u\|_0 = \inf\{v : u \leq v\, u_0, v \in R_+\},$$

(3) the function $u = 0$ is the unique solution of the inequality

$$u \leq K_1u + K_2u + K_3u + Lu$$

in the class $M(D, R_+, u_0)$.

In order to state second lemma we define the sequence $\{u_n\}$ as

$$u_{n+1} = K_1u_n + K_2u_n + K_3u_n + Lu_n, \; n = 0, 1, 2, \ldots \qquad \ldots(2.2)$$

where u_0 is as defined in Lemma 1.

Lemma 2 – If assumptions of Lemma 1 are satisfied then $0 \leq u_{n+1} \leq u_n$, $n = 0, 1, 2 \ldots$, and $u_n \to 0$ for $n \to \infty$ uniformly on D.

To prove our results we need to define the sequence $\{z_n\}$ of random functions as

$$z_{n+1} = Tz_n, \; n = 0, 1, 2 \ldots \qquad \ldots(2.3)$$

where z_0 is arbitrarily fixed element of $C(D, L_2)$ and T is defined by (1.1). Further, for convenience we state the following stochastic integral isometry for two independent variables which is used repeatedly in our proofs.

$$E\left[\left|\int_0^s\int_0^t \phi(u, v, \omega)\, d\beta(u, v, \omega)\right|^2\right] = \int_0^s\int_0^t E[|\phi(u, v, \omega)|^2]\, du dv \qquad \ldots(2.4)$$

see [16, pp. 220-225 and p. 231].

We are now in a position to state the following theorem which yields sufficient conditions for the existence and uniqueness of a solution of equation (1.1).

Theorem 1 — Let assumptions (H_1) and (H_2) and conditions (ii) and (iii) of Lemma 1 hold with k_i, 1, $i = 1, 2, 3$ as defined in (2.1) and

$$r(x,y) = 2E\,[|(Tz_0)\,(x,y,\omega) \in z_0(x,y,\omega)|^2], \text{ for } (x,y) \in D. \quad \ldots(2.5)$$

Then there exists a random solution $\bar{z}(x,y,\omega) \in C(D,L_2)$ of equation (1.1) such that

$$E\,[|\bar{z}(x,y,\omega) - z_n(x,y,\omega)|^2] \leq u_n(x,y),\; n = 0,1,2,\ldots, (x,y) \in D. \quad \ldots(2.6)$$

The solution $\bar{z}$ is unique in the class

$$M(D,L_2,u_0) \overset{df}{=} \{z : z \in M(D,L_2),\; E\,[|z(x,y,\omega) - z_0(x,y,\omega)|^2] \in M(D,R_+,u_0)\}$$

where $M(D,L_2)$ is the class of measurable random functions defined on D, and the class $M(D,R_+,u_0)$ is as defined in Lemma 1.

Remark 1 : It is interesting to note that if we assume both the hypotheses (H_1) and (H_2) hold and

$$k_1(x,y)a(x,y)b(x,y) + k_2(x,y)c(x,y) + k_3(x,y)d(x,y) + 1(x,y) < 1,\; (x,y) \in D$$

then by using the well known Banach fixed point theorem we get the existence and uniqueness of the random solution of equation (1.1).

We next state a theorem concerning the continuous dependence of the solution on the right hand side of equation (1.1). For this we consider another equation

$$v(x,y,\omega) = F^0(x,y,A_1^0\, v,\, A_2^0\, v,\, A_3^0\, v,\, B_1^0\, v,\, B_2^0\, v,\, B_3^0,\, C^0\, v,\, \omega) \quad \ldots(2.7)$$

where

$$A_1^0\, v = \int_0^{a^0(x,y)} \int_0^{b^0(x,y)} f_1^0\,(x,y,s,t,v(s,t,\omega),\,\omega)\, dsdt$$

$$A_2^0\, v = \int_0^{c^0(x,y)} f_2^0\,(x,y,s,v(s,q^0(x,y),\omega),\,\omega)\, ds$$

$$A_3^0\, v = \int_0^{d^0(x,y)} f_3^0\,(x,y,t,v(p^0(x,y),t,\omega),\,\omega)\, dt$$

$$B_1^0\, v = \int_0^{a^0(x,y)} \int_0^{b^0(x,y)} g_1^0\,(x,y,s,t,v(s,t,\omega),\,\omega)\, d\beta(s,t,\omega)$$

$$B_2^0\, v = \int_0^{c^0(x,y)} g_2^0\,(x,y,s,v(s,q^0(x,y),\omega),\,\omega)\, d\beta(s,y,\omega)$$

$$B_3^0 v = \int_0^{d^0(x,y)} g_3^0 (x,y,t,v(p^0(x,y),t,\omega),\omega) \, d\beta(x,t,\omega)$$

$$C^0 v = v(g^0(x,y), h^0(x,y),\omega)$$

and the functions f_i^0, g_i^0 $(i=1, 2, 3)$, F^0, a^0, b^0, c^0, d^0, g^0, h^0, p^0 and q^0 are defined as to those of the corresponding functions f_i, g_i, $(i=1, 2, 3)$, F, a, b, c, d, g, h, p and q respectively.

Let $\bar{v}$ denote a solution of equation (2.7). We take $\phi \in C(D, R_+)$ as

$$\phi(x,y) \stackrel{df}{=} E\,[|(T\bar{v})\ (x,y,\omega) - \bar{v}\ (x,y,\omega)|^2],\ (x,y) \in D$$

where the operator T is as defined by (1.1). Suppose $\psi \in C(D, R_+)$ be such that

$$E\,[|\bar{z}(x,y,\omega) - \bar{v}(x,y,\omega)|^2] \leq \psi(x,y),\ (x,y) \in D.$$

Put

$$r^0(x,y) = \max\,[r(x,y),\ \phi(x,y),\ \psi(x,y)],\ (x,y) \in D$$

where $r(x,y)$ is as defined in (2.5).

Theorem 2 – Let assumptions (H_1) and (H_2) and conditions (ii) and (iii) of Lemma 1 hold with k_i, 1, $i=1, 2, 3$ as defined in (2.1) and $r(x,y)$ replaced by $r^0(x,y)$, $(x,y) \in D$. Then there exists a solution $u^0 \in C(D, R_+)$ of equation

$$u = K_1 u + K_2 u + K_3 u + Lu + \phi \qquad \ldots(2.8)$$

such that

$$E\,|\bar{z}(x,y,\omega) - \bar{v}(x,y,\omega)|^2 \leq u^0(x,y),\ (x,y) \in D. \qquad \ldots(2.9)$$

3. Proofs of Theorems 1 – 2

It can be easily observed from the hypotheses of the theorems and properties of stochastic integral[1,11,16] that integrals in the equation (2.3) for each n exists and $r_n \in C(D,L_2)$, $n = 0, 1, \ldots$.

We shall first establish the following estimations.

$$E\,[|z_n(x,y,\omega) - z_0(x,y,\omega)|^2] \leq u_0(x,y),\ n = 0,1,2,\ldots,\ (x,y) \in D \quad \ldots(3.1)$$

$$E\,[|z_{n+m}(x,y,\omega) - z_n(x,y,\omega)|^2] \leq u_n(x,y),\ n,m = 0,1,2,\ldots,\ (x,y) \in D. \quad \ldots(3.2)$$

It is trivial that (3.1) holds for $n = 0$. If we assume that (3.1) is true for some $n > 0$ then by using the inequality $(x_1 + x_2)^2 \leq 2(x_1^2 + x_2^2)$, hypothesis (H_2) I^0, *the inequality*

$$\left(\sum_{i=1}^{7} x_i\right)^2 \leq 7 \sum_{i=1}^{7} x_i^2$$

Schwarz inequality and stochastic integral isometry for one variable [Arnold[1], p. 66] and by applying (2.4) we obtain

$$E|z_{n+1}(x,y,\omega) - z_0(x,y,\omega)|^2 = E|(Tz_n)(x,y,\omega) - z_0(x,y,\omega)|^2$$

$$= E|(Tz_n)\ (x,y,\omega) - (Tz_0)\ (x,y,\omega) + (Tz_0)\ (x,y,\omega) - z_0(x,y,\omega)|^2$$

$$\leq 2E|(Tz_n)\ (x,y,\omega) - (Tz_0)\ (x,y,\omega)| + 2E|(Tz_0)\ (x,y,\omega) - z_0(x,y,\omega)|^2$$

$$= 2E|F(x,y,A_1z_n,\ A_2z_n,\ A_3z_n,\ B_1z_n,\ B_2z_n,\ B_3z_n,\ Cz_n,\ \omega)$$
$$- F(x,y,A_1z_0,\ A_2z_0,\ A_3z_0,\ B_1z_0,\ B_2z_0,\ B_3z_0,\ Cz_0,\ \omega)|^2$$
$$+ 2E|(Tz_0)\ (x,y,\omega) - z_0(x,y,\omega)|^2$$

$$\leq 2E\ [k_1(x,y)|A_1z_n - A_1z_0| + k_2(x,y)|A_2z_n - A_2z_0|$$
$$+ k_3(x,y)|A_3z_n - A_3z_0| + k_4(x,y)|B_1z_n - B_1z_0|$$
$$+ k_5(x,y)|B_2z_n - B_2z_0| + k_6(x,y)|B_3z_n - B_3z_0|$$
$$+ \bar{l}(x,y)|Cz_n - Cz_0|]^2 + 2E|(Tz_0)(x,y,\omega) - z_0(x,y,\omega)|^2$$

$$\leq 14E\ [k_1^2(x,y)|A_1z_n - A_1z_0|^2 + k_2^2(x,y)|A_2z_n - A_2z_0|^2$$
$$+ k_3^2(x,y)|A_3z_n - A_3z_0|^2 + k_4^2(x,y)|B_1z_n - B_1z_0|^2$$
$$+ k_5^2(x,y)|B_2z_n - B_2z_0|^2 + k_6^2(x,y)|B_3z_n - B_3z_0|^2$$
$$+ \bar{l}^2(x,y)|Cz_n - Cz_0|^2] + 2E|(Tz_0)(x,y,\omega) - z_0(x,y,\omega)|^2$$

$$\leq 14\left[k_1^2(x,y)\ a(x,y)\ b(x,y)\int_0^{a(x,y)}\int_0^{b(x,y)} E|f_1(x,y,s,t,z_n(s,t,\omega),\omega)\right.$$
$$- f_1(x,y,s,t,z_0(s,t,\omega),\omega)|^2\, dsdt$$

$$+ k_2^2(x,y)\ c(x,y)\int_0^{c(x,y)} E|f_2(x,y,s,z_n(s,q(x,y),\omega),\omega)$$
$$- f_2(x,y,s,z_0(s,q(x,y),\omega),\omega)|^2\, ds$$

$$+ k_3^2(x,y)\ d(x,y)\int_0^{d(x,y)} E|f_3(x,y,t,z_n(p(x,y),t,\omega),\omega)$$
$$- f_3(x,y,t,z_0(p(x,y),t,\omega),\omega)|^2\, dt$$

$$+ k_4^2(x,y)\int_0^{a(x,y)}\int_0^{b(x,y)} E|g_1(x,y,s,t,z_n(s,t,\omega),\omega)$$
$$- g_1(x,y,s,t,z_0(s,t,\omega),\omega)|^2\, dsdt$$

$$+ k_5^2(x,y)\int_0^{c(x,y)} E|g_2(x,y,s,z_n(s,q(x,y),\omega),\omega)$$
$$- g_2(x,y,s,z_0(s,q(x,y),\omega),\omega)|^2 ds$$

$$+ k_6^2(x,y) \int_0^{d(x,y)} E|g_3(x,y,t,z_n(p(x,y),t,\omega),\omega) - g_3(x,y,t,z_0(p(x,y),t,\omega),\omega)|^2\, dt$$

$$+ l^2(x,y)\, E|z_n\, g(x,y),\, h(x,y),\omega) - z_0(g(x,y),\, h(x,y),\omega)|^2\Big]$$

$$+ 2E|(Tz_0)\ (x,y,\omega) - z_0(x,y,\omega)|^2$$

$$\leq 14\Big[k_1^2(x,y)\, a(x,y)\, b(x,y)\, m_1^2\,(x,y) \int_0^{a(x,y)}\int_0^{b(x,y)} E|z_n(s,t,\omega) - z_0(s,t,\omega)|^2\, dsdt$$

$$+ k_3^2(x,y)\, c(x,y)\, m_2^2\,(x,y) \int_0^{c(x,y)} E|z_n(s,q(x,y),\omega) - z_0(s,q(x,y),\omega)|^2\, ds$$

$$+ k_3^2(x,y)d(x,y)\, m_3^2\,(x,y) \int_0^{d(x,y)} E|z_n(p(x,y),t,\omega) - z_0(p(x,y),t,\omega)|^2\, dt$$

$$+ k_4^2(x,y)\, m_4^2\,(x,y) \int_0^{a(x,y)}\int_0^{b(x,y)} E|z_n(s,t,\omega) - z_0(s,t,\omega)|^2\, dsdt$$

$$+ k_5^2(x,y)\, m_5^2\,(x,y) \int_0^{c(x,y)} E|z_n(s,q(x,y),\omega) - z_0(s,q(x,y),\omega)|^2\, ds$$

$$+ k_6^2(x,y)\, m_6^2\,(x,y) \int_0^{d(x,y)} E|z_n(p(x,y),t,\omega) - z_0(p(x,y),t,\omega)|^2\, dt$$

$$+ l^2(x,y)\, E|z_n(g(x,y),\, h(x,y),\omega) - z_0(g(x,y),\, h(x,y),\omega)|^2\Big]$$

$$+ 2E|(Tz_0)\ (x,y,\omega) - z_0(x,y,\omega)|^2$$

$$\leq k_1(x,y) \int_0^{a(x,y)}\int_0^{b(x,y)} u_0(s,t)\, dsdt + k_2(x,y) \int_0^{c(x,y)} u_0(s,q(x,y))\, ds$$

$$+ k_3(x,y) \int_0^{d(x,y)} u_0(p(x,y),t)dt + l(x,y)u_0(g(x,y),h(x,y)) + r(x,y)$$

$$= (k_1u_0)(x,y) + (K_2u_0)(x,y) + (K_3u_0)(x,y) + (Lu_0)(x,y)$$

$$+ r(x,y) = u_0(x,y).$$

This completes the proof of (3.1)

In order to prove the inequality (3.2) we use the method of induction once again. It is easy to observe that (3.2) follows from (3.1) for $n = 0$ and $m = 0, 1, 2, \ldots$. Next observing

$$E|z_{n+m+1}(x,y,\omega) - z_{n+1}(x,y,\omega)|^2 = E|(Tz_{n+m})(x,y,\omega) - (Tz_n)(x,y,\omega)|^2$$
$$\leq u_{n+1}(x,y), \ (x,y) \in D$$

and using induction we get (3.2)

From (3.2) and uniform convergence of the sequence $\{u_n\}$ to zero it follows that the sequence $\{z_n\}$ satisfies Cauchy condition and it is uniform in mean square on D, that is convergent to random function $\bar{z} \in C(D,L_2)$. On taking limit as $m \to \infty$ in (3.2) we get (2.6).

It is easy to observe that

$$\begin{aligned} &E|\bar{z}(x,y,\omega) - (T\bar{z})(x,y,\omega)|^2 \\ &= E|\bar{z}(x,y,\omega) - z_n(x,y,\omega) + (Tz_{n-1})(x,y,\omega) - (T\bar{z})(x,y,\omega)|^2 \\ &\leq 2E|\bar{z}(x,y,\omega) - z_n(x,y,\omega)|^2 + 2E|(Tz_{n-1})(x,y,\omega) - (T\bar{z})(x,y,\omega)|^2 \\ &\leq 4u_n(x,y), \ n = 0,1,2,\ldots, \ (x,y) \in D. \end{aligned} \qquad \ldots(3.3)$$

From (3.3) it follows that the random function $\bar{z}$ satisfies equation (1.1).

To prove the uniqueness, let $\hat{z} \in M(D,L_2,u_0)$ be another solution of equation (1.1). It is easy to observe that

$$\hat{u}(x,y) = E\,[|\bar{z}(x,y,\omega) - \hat{z}(x,y,\omega)|^2] \in M(D,R_+,u_0)$$

and

$$\hat{u} \leq \bar{K}_1\hat{u} + K_2\hat{u} + \bar{K}_3\hat{u} + L\hat{u}.$$

Thus, it follows from the assertion (3) of Lemma 1 that

$$E\,[|\bar{z}(x,y,\omega) - \hat{z}(x,y,\omega)|^2] = 0$$

and this completes the proof of Theorem 1.

We construct a sequence $\{u_n^0\}$ such that

$$u_{n+1}^0 = K_1u_n^0 + K_2u_n^0 + K_3u_n^0 + Lu_n^0 + \phi$$

where u_0^0 is the solution of equation (2.8) with ϕ replaced by r^0.

By induction we obtain

$$0 \leq u_{n+1}^0 \leq u_n^0 \leq u_0^0, \ n = 0, 1, 2, \ldots .$$

Thus by Lemma 1 we observe that the sequence $\{u_n^0\}$ is convergent to u^0, $u^0 \leq u_0^0$, which satisfies the equation (2.8). Further by induction we get,

$$E\,[|\bar{z}(x,y,\omega) - \bar{v}(x,y,\omega)|^2] \leq u_n^0(x,y), \ n = 0, 1, 2, \ldots, \ (x,y) \in D. \qquad \ldots(3.4)$$

Now on taking limit as $n \to \infty$ in (3.4) we get (2.9) and the proof of Theorem 2 is complete.

Remark 2 : We note that equation (1.1) is of more general type and in particular if we impose on $a, b, c, d, g, h, p, q, f_i, g_i, i=1, 2, 3$ and F various meanings it is apparent that equation (1.1) has great diversity. For example if we take

$$F(x, y, A_1z, A_2z, A_3z, B_1z, B_2z, B_3z, cz, \omega)$$
$$= f(x,y,cz,\omega) + A_1z + A_2z + A_3z + B_1z + B_2z + B_3z$$

then equation (1.1) reduces to Itô type stochastic integral equation which in turn in the special cases contains the stochastic integral and differential equations in the plane studied by various authors in [10, 12, 16].

In concluding this paper, we note that the method employed in this paper can be very easily extended to study the more general Itô type stochastic functional integral equation in two independent variables of the form

$$\begin{aligned} z(x,y) &= H(x,y,A_1z,\ldots,A_kz,B_1z,\ldots,B_lz,c_1z,\ldots,c_mz, \\ &\quad D_1z,\ldots,D_nz,E_1z,\ldots,E_pz,G_1z,\ldots,G_qz,Pz,\omega) \\ &= (Tz)\ (x,y,\omega) \end{aligned} \qquad \ldots(3.5)$$

where

$$A_iz = \int_0^{a(x,y)} \int_0^{b(x,y)} f_{1i}(x,y,s,t,z(s,t,\omega),\omega)\ dsdt,\ i = 1, 2,\ldots,k,$$

$$B_iz = \int_0^{c(x,y)} f_{2i}(x,y,s,z(s,q(x,y),\omega),\omega)\ ds,\ i = 1, 2,\ldots,l,$$

$$C_iz = \int_0^{d(x,y)} f_{3i}(x,y,t,z(p(x,y),t,\omega),\omega)\ dt,\ i = 1, 2,\ldots,m,$$

$$D_iz = \int_0^{a(x,y)} \int_0^{b(x,y)} g_{1i}(x,y,s,t,z(s,t,\omega),\omega)\ d\beta\ (s,t,\omega),\ i = 1, 2,\ldots,n,$$

$$E_iz = \int_0^{c(x,y)} g_{2i}(x,y,s,z(s,q(x,y),\omega),\omega)\ d\beta\ (s,y,\omega),\ i = 1, 2,\ldots,p,$$

$$G_iz = \int_0^{d(x,y)} g_{3i}(x,y,t,z(p(x,y),t,\omega),\omega)\ d\beta\ (x,t,\omega),\ i = 1, 2,\ldots,q,$$

$$Pz = z(\theta(x,y),\eta(x,y),\omega),$$

under some suitable conditions on the functions involved in (3.5) in view of the conditions involved in the study of equation (1.1). In this case the notation will be much more complicated but the details will be essentially the same. We omit the details.

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