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ON A NONLINEAR STOCHASTIC EVOLUTION EQUATION

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The present paper deals with the study of existence, uniqueness, error estimates and continuous dependence of mild solutions of a nonlinear stochastic evolution equation of the general form in a Hilbert space. The method employed in our analysis is based on the application of Ważewski's general method of successive approximations.

1. INTRODUCTION

This paper deals with properties of mild solutions of stochastic evolution equations of the form

$$\left.\begin{aligned} dz(t) &= A(t)z(t)dt + C(t,z(t))dt + B(t,z(t))dw(t) \\ &\quad + G(t)dm(t) + f(t)dt \\ z(t_0) &= z_0 \end{aligned}\right\} \quad \ldots(1.1)$$

in a real separable Hilbert space H. Throughout this paper we suppose that $(\Omega, \mathfrak{F}, P)$ is a complete probability space, H, K are real separable Hilbert spaces, $I = [t_0, T]$, $t_0 \geq 0$ and $\mathfrak{L}(K, H)$ denotes the class of bounded linear operators from K to H and we shall write stochastic processes and functions by suppressing the argument ω, $\omega \in \Omega$ without further mention. In (1.1), $A(t)$ is a linear possibly unbounded operator on H which generated a strong evolution operator $U(t, s)$ on H(see Arnold *et al.*[1]), $w(t)$ and $m(t)$ are K_1-valued and K_2-valued Wiener process and stationary independent increment process (s.i.i.p.) respectively, K_1, K_2 are real separable Hilbert spaces, $z_0 \in L_2(\Omega, P; H)$, $\mathfrak{F}_0$-measurable and $E\|z_0\|_H^2 < \infty$, $C(t, x) : I \times H \to H$, is measurable in (t, x),

$$\int_{t_0}^{T} E\|C(t, x)\|_H^2 \, dt < \infty,$$

$$B(t, x) : I \times H \to \mathfrak{L}(K_1, H)$$

is measurable in (t, x),

$$\int_{t_0}^{T} E\|B(\tau, x)\|^2_{\mathcal{L}(K_1,H)}\, dt < \infty,$$

$G(t) : I \to \mathcal{L}(K_2, H)$, $\mathcal{F}_t$-adapted processes ϕ, measurable in t such that

$$\int_{t_0}^{T} E\|\phi(t)\|^2_{\mathcal{L}(K_2,H)}\, dt < \infty,$$

$f(t)$ is an H-valued stochastic process adapted to $\mathcal{F}_t$,

$$\int_{t_0}^{T} E\|f(t)\|^2_H\, dt < \infty$$

and $z \in C([t_0, T], L_2(\Omega, P; H))$, H-valued stochastic processes continuous in mean square, $\mathcal{F}_t$-adapted and measurable in t.

Stochastic evolution equations of type (1.1) have been considered by Curtain[2], Curtain and Flab[3], DaPrato *et al.*[4], Ichikawa[5,6], Kuo[7], Tubaro[9], Tudor[10] and by many authors. In particular in a technical report, published in 1980, Arnold *et al.*[1] have studied the existence and uniqueness properties of mild solutions of eqn. (1.1) by using Banach contraction principle. In this paper we shall study the existence, uniqueness, error estimation and continuous dependence of the mild solutions of eqn. (1.1). Our approach to the problem is different from the above mentioned authors and is based on the application of Ważewski's general method of successive approximations[12] (see, also Pachpatte[8] and Turo[11]).

2. Statements of Results

Before giving our main results we introduce the definition of s.i.i.p. and properties of centered s.i.i.p. from Arnold *et al.*[1] (see, also Curtain[2]).

We define $\{m(t), t \in I\}$ as a s.i.i.p. on K if $m(t) \in L_2(\Omega, P; K)$ for all $t \in I$ and

$$E\{m(t)\} = kt, \text{ for some } k \in K \qquad \ldots(2.1)$$

$$\text{Cov}\{m(t) - m(s)\} = M(t - s), \text{ for some } M \in \mathcal{L}(K) \qquad \ldots(2.2)$$

nuclear, positive and self adjoint where $\mathcal{L}(K)$ denotes the class of bounded linear operators from K to K, and for $t_2 > t_1$, $m(t_2) - m(t_1)$ is independent of $\sigma(m(s), s \le t_1)$ i.e. of the σ-algebra generated by $m(s)$, $s \le t_1$. We further quote that for $\phi(\cdot) \in B_2(I, \mathcal{L}(K, H))$, the class of $\mathcal{L}(K, H)$-valued, $\mathcal{F}_t$-adapted processes ϕ, measurable in t such that

$$\int_{t_0}^{t} E\|\phi(t)\|^2_{\mathcal{L}(K,H)}\, dt < \infty \text{ and } m(t) \text{ as s.i.i.p.}$$

$$y(t) = \int_{t_0}^{t} \phi(s)\, dm(s)$$

is well defined stochastic process on H. A centered s.i.i.p. satisfies the following properties.

$$y(t) \in C(I, L_2(\Omega, P; K)) \qquad \ldots(2.3)$$

$$E\left\{ \int_{t_0}^{t} \phi(s)\, dm(s) \right\} = 0 \qquad \ldots(2.4)$$

$$E\left\{ \| \int_{t_0}^{t} \phi(s)\, dm(s) \|^2 \right\} = \text{trace } M \int_{t_0}^{t} E\|\phi(s)\|^2\, ds \qquad \ldots(2.5)$$

where M is defined by (2.2).

A K-valued Wiener process is a centered K-valued orthogonal increments process on I given by

$$w(t) = \sum_{i=0}^{\infty} \mu_i(t)\, e_i$$

where $\{e_i\}$ is a complete orthonormal basis for K and $\mu_i(t)$ are mutually independent real Wiener processes with

$$E\{\mu_i(t)^2\} = t\lambda_i \text{ and } \sum_{i=0}^{\infty} \lambda_i < \infty,\ \lambda_i \geq 0.$$

So

$$E\{w(t)\} = 0$$

$$\text{Cov}\{w(t) - w(s)\} = E\{(w(t) - w(s)) \circ (w(t) - w(s))\}$$

$$= W(t - s),\ 0 \leq s \leq t \leq T$$

where 0 is a tensor product on K defined by $(h \circ k)\, x = h < k, x >$, $h, k \in K$ and W is a positive, nuclear operator with $We_i = \lambda_i e_i$. Further, we need the following stochastic integral isometry corresponding to K-valued Wiener process which is used repeatedly in our proofs.

$$E\left\{ \| \int_{t_0}^{t} \phi(s)\, dw(s) \|^2 \right\} \leq \text{trace } W \int_{t_0}^{t} E\|\phi(s)\|^2\, ds. \qquad \ldots(2.6)$$

For the definitions of mild evolution operator and strong evolution operator and their various properties we refer Arnold *et al.*[1], (also see Curtain[2]). Throughout our discussion we take $U(t, s)$ as an $\mathcal{L}(H)$-valued mild evolution operator defined on $\Delta(t_0, T)$ where

$$\Delta(t_0, T) = \{(t, s) : o \leq t_0 \leq s \leq t \leq T\}.$$

A stochastic process $z(t)$, $t \in I$, is a mild solution of (1.1) if $z(t)$ is adapted to $\mathcal{F}_t$, measurable in t and

$\int_{t_0}^{T} E\|z(s)\|^2\,ds < \infty$ satisfies the equation

$$z(t) = U(t, t_0)\,z_0 + \int_{t_0}^{t} U(t, s)\,C(s, z(s))\,ds + \int_{t_0}^{t} U(t, s)\,B(s, z(s))\,dw(s) + \int_{t_0}^{t} U(t, s)\,G(s)\,dm(s) + \int_{t_0}^{t} U(t, s)\,f(s)\,ds \quad \ldots(2.7)$$

for all $t \in I$ w.p. 1.

By making formal substitutions

$$u(t) = C(t, z(t)),\ v(t) = B(t, z(t)) \quad \ldots(2.8)$$

in (2.7), it is easy to observe that (2.7) is equivalent to the following pair of equations

$$u(t) = C\left(t, U(t, t_0)\,z_0 + \int_{t_0}^{t} U(t, s)u(s)ds + \int_{t_0}^{t} U(t, s)v(s)\,dw(s) + \int_{t_0}^{t} U(t, s)G(s)dm(s) + \int_{t_0}^{t} U(t, s)f(s)ds\right) = T_1\,[u,v]\,(t) \quad \ldots(2.9)$$

$$v(t) = B\left(t, U(t, t_0)\,z_0 + \int_{t_0}^{t} U(t, s)u(s)ds + \int_{t_0}^{t} U(t, s)v(s)\,dw(s) + \int_{t_0}^{t} U(t, s)G(s)dm(s) + \int_{t_0}^{t} U(t, s)f(s)ds\right) = T_2\,[u,v]\,(t) \quad \ldots(2.10)$$

for $t \in I$.

By a solution of the system (2.9) – (2.10) we mean a pair $(u(t), v(t))$ such that

(i) $u(t) : I \times H \to H$ is continuous in mean square, $\mathfrak{F}_t$-adapted and measurable in t,

(ii) $v(t) : I \times H \to \mathfrak{L}(K_1, H)$ is $\mathfrak{F}_t$-adapted, measurable in t such that

$$\int_{t_0}^{T} E\|v(t)\|^2_{\mathfrak{L}(K_1,H)}\,dt < \infty$$

and satisfies the system (2.9) – (2.10).

The problem of solving the integral equation (2.7) is equivalent to the problem of solving eqns. (2.9) – (2.10) and hence we will deal with the equations (2.9) – (2.10) in our subsequent discussion.

Next we list the following assumptions used throughout this paper :

(A_1) Suppose that there exists a Hilbert space V_1 such that $V_1 \subseteq H$ with dense continuous injections and that

$$U(t, s) \in \mathcal{L}(H, V_1) \text{ for } t > s \quad \text{and}$$

$$\|U(t, s)\, h\|_{V_1} \leq g(t - s)\|h\|_H,\ h \in H,\ g \in L_2\ (0, T - t_0).$$

(A_2) For $h_i \in H$, $i = 1, 2$,

(i) there exists a constant $L_1 > 0$ such that

$$\|C(t, h_1) - C(t, h_2)\|_H^2 \leq L_1 \|h_1 - h_2\|_H^2$$

(ii) there exists a constant $L_2 > 0$ such that

$$\|B(t, h_1) - B(t, h_2)\|_{\mathcal{L}(K_1, H)}^2 \leq L_2 \|h_1 - h_2\|_H^2.$$

(A_3) Suppose that there exists a pair of continuous functions $(\bar{x}, \bar{y})$, $\bar{x}, \bar{y} : I \to R_+ = [0, \infty)$ satisfying the inequalities

$$\left.\begin{aligned} M_1[x, y](t) + r_1(t) &\leq \dot{x}(t) \\ M_2[x, y](t) + r_2(t) &\leq y(t) \end{aligned}\right\} \qquad \ldots(2.11)$$

where

$$M_1[x, y](t) = 4L_1 d^2 \left[T \int_{t_0}^{t} x(s)ds + \text{trace } W \int_{t_0}^{t} y(s)ds \right] \qquad \ldots(2.12)$$

$$M_2[x, y](t) = 4L_2 d^2 \left[T \int_{t_0}^{t} x(s)ds + \text{trace } W \int_{t_0}^{t} y(s)ds \right] \qquad \ldots(2.13)$$

$$r_1(t) = 12L_1 d^2 \sup_{t_0 \leq t \leq T} \left[1 + \text{trace } M \int_0^T E\|G(t)\|_{\mathcal{L}(K_2, H)}^2\, dt + T \int_{t_0}^{T} E\|f(t)\|_H^2\, ds \right] + 4 \sup_{t_0 \leq t \leq T} E\|C(t, 0)\|_H^2 \qquad \ldots(2.14)$$

$$r_2(t) = 12L_2 d^2 \sup_{t_0 \leq t \leq T} \left[1 + \text{trace } M \int_{t_0}^T E\|G(t)\|_{\mathcal{L}(K_2, H)}^2\, dt + T \int_{t_0}^{T} E\|f(t)\|_H^2\, dt \right] + 4 \sup_{t_0 \leq t \leq T} E\|B(t, 0)\|_{\mathcal{L}(K_1, H)}^2 \qquad \ldots(2.15)$$

and $d > 0$, $l > 0$ are constants and L_1, L_2 are as defined in the assumption (A_2).

(A_4) In the class of functions satisfying $0 \leq x(t) \leq \bar{x}(t)$, $0 \leq y(t) \leq \bar{y}(t)$, $t \in I$, the pair of functions $(x(t), y(t)) \equiv (0,0)$ is the only measurable solution of the equations

$$\left.\begin{aligned} x(t) &= M_1 [x, y] (t) \\ y(t) &= M_2 [x, y] (t) \end{aligned}\right] \quad \ldots(2.16)$$

where M_1 and M_2 are as defined in (2.12) and (2.13) respectively.

To prove the existence of a solution of the system (2.9) – (2.10) we define the sequence $\{(u_m, v_m)\}$, $u_m : I \times H \to H$, $v_m : I \times H \to \mathcal{L}(K_1, H)$ of random functions in I as

$$\left.\begin{aligned} u_0(t) &\equiv 0, \quad u_{m+1}(t) = T_1 [u_m, v_m] (t) \\ v_0(t) &\equiv 0, \quad v_{m+1}(t) = T_2 [u_m, v_m] (t) \end{aligned}\right\} \quad \ldots(2.17)$$

for $t \in I$, $m = 0, 1, 2, \ldots$ and T_1 and T_2 are defined by (2.9) and (2.10) respectively. Further in order to prove the convergence of the sequence $\{(u_m, v_m)\}$ to the solution $(\bar{u}, \bar{v})$ of (2.9) – (2.10) we define a sequence $\{(x_m, y_m)\}$, $x_m, y_m : I \to R_+$ by the relations

$$\left.\begin{aligned} x_0(t) &\equiv \bar{x}(t), \quad x_{m+1}(t) = M_1 [x_m, y_m] (t) \\ y_0(t) &\equiv \bar{y}(t), \quad y_{m+1}(t) = M_2 [x_m, y_m] (t) \end{aligned}\right] \quad \ldots(2.18)$$

for $t \in I$, $m = 0, 1, 2, \ldots$ and M_1 and M_2 are defined by (2.12) and (2.13) respectively.

We prepare the following lemma needed in the proofs of our main results. For similar lemma see Pachpatte[8].

Lemma – Suppose that the assumptions $(A_3) - (A_4)$ hold. Then

$$0 \leq x_{m+1}(t) \leq x_m(t) \leq \bar{x}(t), \; 0 \leq y_{m+1}(t) \leq y_m(t) \leq \bar{y}(t) \quad \ldots(2.19)$$

for $t \in I$, $m = 0, 1, 2, \ldots$, and $x_m \to 0$, $y_m \to 0$ for $m \to \infty$ uniformly in I.

The relations in (2.19) follows by induction arguments. Further the relations in (2.19) implies the convergence of the sequence $\{(x_m, y_m)\}$. The limit of this sequence satisfies the system (2.16) and by assumption (A_4) it must be identically equal to zero. The uniform convergence of $\{(x_m, y_m)\}$ follows by Dini's theorem.

Now we state the following theorem which yields sufficient conditions for the existence and uniqueness of a solution of the system (2.9) – (2.10).

Theorem 1 – Let assumptions $(A_1) - (A_4)$ hold and $r_1(t)$ and $r_2(t)$, for $t \in I$ are as defined in (2.14) and (2.15) respectively. Then there exists random solution $(\bar{u}(t), \bar{v}(t))$, $\bar{u} : I \times H \to H$, $\bar{v} : I \times H \to \mathcal{L}(K_1, H)$ of the system (2.9) – (2.10) such that the following estimations :

$$\left.\begin{aligned} E[\|\bar{u}(t) - u_m(t)\|_H^2] &\leq x_m(t) \\ E[\|\bar{v}(t) - v_m(t)\|^2_{\mathcal{L}(K_1,H)}] &\leq y_m(t) \end{aligned}\right\} \quad \ldots(2.20)$$

and

$$\left.\begin{aligned} E[\|\bar{u}(t)\|_H^2] &\leq \bar{x}(t) \\ E[\|\bar{v}(t)\|^2_{\mathcal{L}(K_1,H)}] &\leq \bar{y}(t) \end{aligned}\right\} \quad \ldots(2.21)$$

hold for $t \in I$, $m = 0, 1, 2, \ldots$, where the sequence $\{(x_m, y_m)\}$ is defined by (2.18). Moreover, the solution $(\bar{u}, \bar{v})$ of eqns. (2.9) – (2.10) is unique in the class of functions satisfying (2.21).

We next state a theorem concerning the continuous dependence of the solution on the right-hand side of eqns. (2.9) – (2.10). For this we consider another pair of equations

$$p(t) = c\left(t, U(t,t_0)\, z_0 + \int_{t_0}^{t} U(t,s)p(s)ds + \int_{t_0}^{t} U(t,s)q(s)dw(s) + \int_{t_0}^{t} U(t,s)G(s)dm(s) + \int_{t_0}^{t} U(t,s)f(s)ds\right) \quad \ldots(2.22)$$

$$q(t) = b\left(t, U(t,t_0)\, z_0 + \int_{t_0}^{t} U(t,s)p(s)ds + \int_{t_0}^{t} U(t,s)q(s)dw(s) + \int_{t_0}^{t} U(t,s)G(s)dm(s) + \int_{t_0}^{t} U(t,s)f(s)ds\right) \quad \ldots(2.23)$$

where the functions p, q, c, b are defined as to those of the corresponding functions u, v, C, B respectively in the system (2.9) – (2.10).

Let $(\bar{p}, \bar{q})$ denote a solution of eqns. (2.22) – (2.23). We take (ϕ_1, ϕ_2), $\phi_1, \phi_2 : I \to R_+$ as

$$\phi_1(t) \overset{df}{=} E\left[\|T_1\,[\bar{p}, \bar{q}]\,(t) - \bar{p}(t)\|_H^2\right]$$

$$\phi_2(t) \overset{df}{=} E\left[\|T_2\,[\bar{p}, \bar{q}]\,(t) - \bar{q}(t)\|_{\mathfrak{L}(K_1,H)}^2\right]$$

for $t \in I$ where the operators T_1 and T_2 are as defined by (2.9) and (2.10) respectively. Suppose $\psi_1, \psi_2 : I \to R_+$ are such that

$$E\left[\|\bar{u}(t) - \bar{p}(t)\|_H^2\right] \leq \psi_1(t)$$

$$E\left[\|\bar{v}(t) - \bar{q}(t)\|_{\mathfrak{L}(K_1,H)}^2\right] \leq \psi_2(t)$$

for $t \in I$. Put

$$\bar{r}_1(t) = \max\left[r_1(t), \phi_1(t), \psi_1(t)\right]$$

$$\bar{r}_2(t) = \max\left[r_2(t), \phi_2(t), \psi_2(t)\right]$$

for $t \in I$ where $r_1(t)$ and $r_2(t)$ are as defined in (2.14) and (2.15) respectively.

Theorem 2 – Let assumptions (A_1) – (A_4) hold and $(r_1(t), r_2(t))$ be replaced by $(\bar{r}_1(t), \bar{r}_2(t))$, $t \in I$. Then there exists a solution $(\bar{\alpha}, \bar{\beta})$, $\bar{\alpha}, \bar{\beta} : I \to R_+$ of the system

$$\left.\begin{aligned}\alpha(t) &= M_1\,[\alpha,\,\beta]\,(t) + \phi_1\,(t)\\ \beta(t) &= M_2\,[\alpha,\,\beta]\,(t) + \phi_2\,(t)\end{aligned}\right\} \qquad \ldots(2.24)$$

where M_1, M_2 are as defined by (2.12), (2.13) respectively, such that

$$\left.\begin{aligned}E\,[\|\bar{u}(t) - \bar{p}(t)\|_H^2] &\le \bar{\alpha}(t)\\ E\,[\|\bar{v}(t) - \bar{q}(t)\|_{\mathfrak{L}(K_1,H)}^2] &\le \bar{\beta}(t)\end{aligned}\right\} \qquad \ldots(2.25)$$

for $t \in I$.

3. Proofs of Theorems 1 – 2

In order to prove Theorem 1, first we establish the following estimations

$$\left.\begin{aligned}E\,[\|u_m(t)\|_H^2] &\le \bar{x}(t)\\ E\,[\|v_m(t)\|_{\mathfrak{L}(K_1,H)}^2] &\le \bar{y}(t)\end{aligned}\right\} \qquad \ldots(3.1)$$

for $m = 0, 1, 2, \ldots, t \in I$, and

$$\left.\begin{aligned}E\,[\|u_{m+n}(t) - u_m(t)\|_H^2] &\le x_m(t)\\ E\,[\|v_{m+n}(t) - v_m(t)\|_{\mathfrak{L}(K_1,H)}^2] &\le y_m(t)\end{aligned}\right\} \qquad \ldots(3.2)$$

for $m, n = 0, 1, 2, \ldots$, and $t \in I$.

It is trivial from (2.17) – (2.18) that (3.1) holds for $m = 0$. Furthermore, if we assume that (3.1) is true for $m \ge 0$ then by using (2.17), inequality $(x_1 + x_2)^2 \le 2(x_1^2 + x_2^2)$, assumptions (A_1) - (A_2) (i), inequality $(x_1 + x_2 + x_3)^2 \le 3(x_1^2 + x_2^2 + x_3^2)$, Schwarz inequality, (2.5) – (2.6) and (2.11), we obtain

$$E\,\|u_{m+1}(t)\|_H^2 = E\|T_1\,[u_m,\,v_m]\,(t) - T_1\,[u_0,\,v_0]\,(t) + T_1\,[u_0,\,v_0]\,(t)\|_H^2$$

$$\le 2E\|T_1\,[u_m,\,v_m]\,(t) - T_1\,[u_0,\,v_0]\,(t)\|_H^2 + 2E\,\|T_1\,[u_0,\,v_0]\,(t)\|_H^2$$

$$\le 2L_1E\|\int_{t_0}^{t} U(t,\,s)\,[u_m(s) - u_0(s)]\,ds$$

$$+ \int_{t_0}^{t} U(t,\,s)\,[v_m(s) - v_0(s)]\,dw(s)\|_H^2$$

$$+ 2E\,\|C\Big(t, U(t,\,t_0)\,z_0 + \int_{t_0}^{t} U(t,\,s)G(s)dm(s) + \int_{t_0}^{t} U(t,\,s)f(s)ds\Big)$$

$$- C(t,\,0) + C(t,\,0)\|_H^2$$

$$\leq 4L_1 \left[E\| \int_{t_0}^{t} U(t, s)u_m(s)ds\|_H^2 + E\| \int_{t_0}^{t} U(t, s)v_m(s)dw(s)\|_H^2 \right]$$

$$+ 4E\| C\left(t, U(t, t_0)z_0 + \int_{t_0}^{t} U(t, s)G(s)dm(s) + \int_{t_0}^{t} U(t,s)f(s)ds\right)$$

$$- C(t, 0)\|_H^2 + 4E\|C(t, 0)\|_H^2$$

$$\leq 4L_1 \left[T \int_{t_0}^{t} E\|U(t, s)u_m(s)\|_H^2 \, ds + \text{trace } W \int_{t_0}^{t} E\|U(t, s)v_m(s)\|_H^2 \, ds \right]$$

$$+ 4L_1 \left(\|U(t, t_0)z_0 + \int_{t_0}^{t} U(t, s)G(s)dm(s) + \int_{t_0}^{t} U(t,s)f(s)ds\|_H^2 \right)$$

$$+ 4E\|C(t, 0)\|_H^2$$

$$\leq 4L_1 \left[T \int_{t_0}^{t} g^2(t - s)E\|u_m(s)\|_H^2 \, ds + \text{trace } W \int_{t_0}^{t} g^2(t - s)E\|v_m(s)\|_{\mathfrak{L}(K_1,H)}^2 \, ds \right]$$

$$+ 12L_1 \left[E\|U(t, t_0)z_0\|_H^2 + \|\int_{t_0}^{t} U(t, s)G(s)dm(s)\|_H^2 \right.$$

$$\left. + E\| \int_{t_0}^{t} U(t,s)f(s)ds\|_H^2 \right] + 4E\|C(t, 0)\|_H^2$$

$$\leq 4L_1 d^2 \left[T \int_{t_0}^{t} E\|u_m(s)\|_H^2 \, ds + \text{trace } W \int_{t_0}^{t} E\|v_m(s)\|_{\mathfrak{L}(K_1,H)}^2 \, ds \right]$$

$$+ 12L_1 \left[g^2(t - t_0)E\|z_0\|_H^2 + \text{trace } M \int_{t_0}^{t} g^2(t - s)E\|G(s)\|_{\mathfrak{L}(K_2,H)}^2 \, ds \right.$$

$$\left. + T \int_{t_0}^{t} g^2(t - s)E\|f(s)\|_H^2 \, ds \right] + 4E\|C(t, 0)\|_H^2$$

$$\leq 4L_1d^2\left[T\int_{t_0}^{t}\bar{x}(s)ds + \text{trace } W\int_{t_0}^{t}\bar{y}(s)ds\right]$$

$$+ 12L_1d^2\left[l + \text{trace } M\int_{t_0}^{t}E\|G(s)\|^2_{\mathfrak{L}(K_2,H)}\,ds\right.$$

$$\left.+ T\int_{t_0}^{t}E\|f(s)\|^2_H\,ds\right] + 4E\|C(t,0)\|^2_H$$

$$\leq M_1[\bar{x},\bar{y}](t) + r_1(t)$$

$$\leq \bar{x}(t),\ t \in I$$

where $\sup_{t_0 \leq s \leq t \leq T} g^2(t-s) = d^2$ since $g \in L_2(0, T)$ and $E\|z_0\|^2_H \leq l$ since $E\|z_0\|^2_H < \infty$.

The details of the proof of second inequality in (3.1) can be easily obtained by following above argument after making use of the assumption (A_2) (ii) in place of (A_2) (i) with suitable changes.

In order to prove (3.2) we make use of the method of induction once again. It is easy to see that (3.2) follows from (3.1) for $m = 0$, $n = 0, 1, 2,\ldots$

$$E\|u_n(t) - u_0(t)\|^2_H = E\|u_n(t)\|^2_H \leq \bar{x}(t) = x_0(t)$$

and

$$E\|v_n(t) - v_0(t)\|^2_{\mathfrak{L}(K_1,H)} = E\|v_n(t)\|^2_{\mathfrak{L}(K_1,H)} \leq \bar{y}(t) = y_0(t)$$

for $t \in I$. Suppose (3.2) is true for $m, n \geq 0$, then by using assumptions $(A_1)-(A_2)$ (i), the inequality $(x_1 + x_2)^2 \leq 2(x_1^2 + x_2^2)$, Schwarz inequality, (2.5)–(2.6) and (2.18) we get

$$E\|u_{m+n+1}(t) - u_{m+1}(t)\|^2_H$$

$$= E\|T_1[u_{m+n}, v_{m+n}](t) - T_1[u_m, v_m](t)\|^2_H$$

$$\leq L_1E\|\int_{t_0}^{t}U(t,s)(u_{m+n}(s) - u_m(s))\,ds$$

$$+ \int_{t_0}^{t}U(t,s)(v_{m+n}(s) - v_m(s))\,dw(s)\|^2_H$$

$$\leq 2L_1\left[E\|\int_{t_0}^{t}U(t,s)(u_{m+n}(s) - u_m(s))\,ds\|^2_H\right.$$

$$+ E\left\| \int_{t_0}^{t} U(t, s)\ (v_{m+n}(s) - v_m(s))\ dw(s) \right\|_H^2 \Bigg]$$

$$\leq 2L_1 \Bigg[T \int_{t_0}^{t} E\|U(t, s)\ (u_{m+n}(s) - u_m(s))\|_H^2\ ds$$

$$+ \text{trace } W \int_{t_0}^{t} E\|U(t, s)\ (v_{m+n}(s) - v_m(s))\|_H^2\ ds \Bigg]$$

$$\leq 2L_1 \Bigg[T \int_{t_0}^{t} g^2(t - s)\ E\|u_{m+n}(s) - u_m(s)\|_H^2\ ds$$

$$+ \text{trace } W \int_{t_0}^{t} g^2(t - s)\ E\|v_{m+n}(s) - v_m(s)\|_{\mathfrak{L}(K_1, H)}^2\ ds \Bigg]$$

$$\leq 2L_1 d^2 \Bigg[T \int_{t_0}^{t} E\|u_{m+n}(s) - u_m(s)\|_H^2\ ds$$

$$+ \text{trace } W \int_{t_0}^{t} E\|v_{m+n}(s) - v_m(s)\|_{\mathfrak{L}(K_1, H)}^2\ ds \Bigg]$$

$$\leq 2L_1 d^2 \left[T \int_{t_0}^{t} x_m(s)\ ds + \text{trace } W \int_{t_0}^{t} y_m(s)\ ds \right]$$

$$\leq M_1\ [x_m, y_m]\ (t) = x_{m+1}\ (t),\ t \in I$$

where $\sup_{t_0 \leq s \leq t \leq T} g^2(t - s) = d^2$ since $g \in L_2\ (0, T)$.

The details of the proof of second inequality in (3.2) can be easily obtained by following above argument after making use of the assumption (A_2) (ii) in place of (A_2) (i) with suitable changes.

From (3.2) and the uniform convergence of the sequence $\{(x_m, y_m)\}$ to zero it follows that the sequence $\{(u_m, v_m)\}$ is uniform in mean square in I, that is, it converges to $(\bar{u}, \bar{v})$. Taking limit as $m \to \infty$ in (3.2) we get (2.20).

By using the inequality $(x_1 + x_2)^2 \leq 2\ (x_1^2 + x_2^2)$, assumptions $(A_1) - (A_2)$, Schwarz inequality, (2.5) – (2.6), (2.18) and (2.20) we get

$$
\begin{aligned}
&E\|\bar{u}(t) - T_1[\bar{u}, \bar{v}](t)\|_H^2 \\
&= E\|\bar{u}(t) - u_m(t) + u_m(t) - T_1[\bar{u}, \bar{v}](t)\|_H^2 \\
&\le 2E\|\bar{u}(t) - u_m(t)\|_H^2 + 2E\|T_1[u_{m-1}, v_{m-1}](t) - T_1[\bar{u}, \bar{v}](t)\|_H^2 \\
&\le 2x_m(t) + 2L_1 E\Big\| \int_{t_0}^{t} U(t, s)\,(u_{m-1}(s) - \bar{u}(s))\,ds \\
&\qquad + \int_{t_0}^{t} U(t, s)\,(v_{m-1}(s) - \bar{v}(s))\,dw(s)\Big\|_H^2 \\
&\le 2x_m(t) + 4L_1\Big[E\Big\| \int_{t_0}^{t} U(t, s)\,(u_{m-1}(s) - \bar{u}(s))\,ds\Big\|_H^2 \\
&\qquad + E\Big\| \int_{t_0}^{t} U(t, s)\,(v_{m-1}(s) - \bar{v}(s))\,dw(s)\Big\|_H^2\Big] \\
&\le 2x_m(t) + 4L_1\Big[T\int_{t_0}^{t} E\|U(t, s)\,(u_{m-1}(s) - \bar{u}(s))\|_H^2\,ds \\
&\qquad + \text{trace } W\int_{t_0}^{t} E\|U(t, s)\,(v_{m-1}(s) - \bar{v}(s))\|_H^2\,ds\Big] \\
&\le 2x_m(t) + 4L_1\Big[T\int_{t_0}^{t} g^2(t - s)\,E\|u_{m-1}(s) - \bar{u}(s)\|_H^2\,ds \\
&\qquad + \text{trace } W\int_{t_0}^{t} g^2(t - s)\,E\|v_{m-1}(s) - \bar{v}(s)\|_{\mathfrak{L}(K_1, H)}^2\,ds\Big] \\
&\le 2x_m(t) + 4L_1 d^2\Big[T\int_{t_0}^{t} x_{m-1}(s)\,ds + \text{trace } W\int_{t_0}^{t} y_{m-1}(s)\,ds\Big] \\
&= 2x_m(t) + M_1[x_{m-1}, y_{m-1}](t) = 2x_m(t) + x_m(t) = 3x_m(t) \qquad \ldots(3.3)
\end{aligned}
$$

for $t \in I$, where $\sup_{t_0 \le s \le t \le T} g^2(t - s) = d^2$ since $g \in L_2(0, T)$. By following the similar argument once again with suitable changes we have

$$E\|\bar{v}(t) - T_2[\bar{u}, \bar{v}](t)\|^2_{\mathfrak{L}(K_1,H)} \leq 3y_m(t) \qquad \ldots(3.4)$$

$m = 0, 1, 2, \ldots$ and $t \in I$.

From (3.3) – (3.4) it follows that the random function $(\bar{u}, \bar{v})$ satisfies the pair of equations (2.9) – (2.10).

To prove the uniqueness, let $(\hat{u}, \hat{v})$ be another solution of the system (2.9) – (2.10) such that

$$\hat{u}(t) \neq \bar{u}(t), \ \hat{v}(t) \neq \bar{v}(t)$$

and

$$E\|\hat{u}(t)\|^2_H \leq \bar{x}(t), \ E\|\hat{v}(t)\|^2_{\mathfrak{L}(K_1,H)} \leq y(t)$$

for $t \in I$, by induction we get

$$E\|\hat{u}(t) - u_m(t)\|^2_H \leq x_m(t), \ E\|\hat{v}(t) - v_m(t)E\|^2_{\mathfrak{L}(K_1,H)} \leq y_m(t)$$

for $m = 0, 1, 2, \ldots$ and $t \in I$. Hence it follows that $\bar{u}(t) \equiv \hat{u}(t)$, $\bar{v}(t) \equiv \hat{v}(t)$, for $t \in I$. This contradiction proves the uniqueness of $(\bar{u}, \bar{v})$ in the class of functions satisfying the relation (2.21). This completes the proof of Theorem 1.

In order to prove Theorem 2 we construct a sequence $\{(\alpha_m, \beta_m)\}$ such that for $t \in I$ and $m = 0, 1, 2, \ldots$

$$\alpha_{m+1}(t) = M_1[\alpha_m, \beta_m](t) + \phi_1(t)$$

$$\beta_{m+1}(t) = M_2[\alpha_m, \beta_m](t) + \phi_2(t)$$

where $(\alpha_0(t), \beta_0(t))$ is the solution of the system

$$\alpha(t) = M_1[\alpha, \beta](t) + \bar{r}_1(t)$$

$$\beta(t) = M_2[\alpha, \beta](t) + \bar{r}_2(t).$$

By induction we obtain

$$0 \leq \alpha_{m+1}(t) \leq \alpha_m(t) \leq \alpha_0(t)$$

$$0 \leq \beta_{m+1}(t) \leq \beta_m(t) \leq \beta_0(t)$$

$m = 0, 1, 2, \ldots$, $t \in I$. Thus by lemma we observe that the sequence $\{(\alpha_m, \beta_m)\}$ is convergent to $(\bar{\alpha}, \bar{\beta})$, $\bar{\alpha} \leq \alpha_0$, $\bar{\beta} \leq \beta_0$, which satisfies the system (2.25). Further by induction we get

$$\left.\begin{aligned} E\|\bar{u}(t) - p(t)\|^2_H &\leq \alpha_m(t) \\ E\|\bar{v}(t) - q(t)\|^2_{\mathfrak{L}(K_1,H)} &\leq \beta_m(t) \end{aligned}\right\} \qquad \ldots(3.5)$$

for $m = 0, 1, 2, \ldots$, and $t \in I$. Now on taking limit as $m \to \infty$ in (3.5) we get (2.25) and the proof of Theorem 2 is complete.

4. An Application

In this section we consider the example given in Arnold *et al.*[1] (p. 17) to illustrate the application of our result established in Theorem 1.

Consider the following nonlinear equation

$$\left.\begin{aligned} dz(t, x) &= \frac{\partial^2}{\partial x^2} z(t, x)\, dt + a(t, x)\, z(t, x)\, dt \\ &\quad + \gamma\,(t, x)\, \sqrt{1 + z^2(t, x)}\, d\beta(t) \\ z(t, 0) &= 0 = z(t, 1) \\ z(0, x) &= z_0\,(x) \end{aligned}\right\} \qquad \ldots(4.1)$$

where $\beta(t)$ is a real scalar Wiener process in $H = L_2\,(0, 1)$ and

$$A = \frac{d^2}{dt^2}$$

with

$$D(A) = \{h \in H;\ h_x,\ h_{xx} \in H,\ h(0) = 0 = h(1)\}$$

generates an analytic semigroup $T(t) : H \to D(A)$ given by

$$T(t)h = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < h,\ e_n >_H e_n$$

$< , >_H$ denotes the inner product on H, $a(t, x)$ and $\gamma(t, x)$ are continuous in (t, x) and uniformly bounded on $[0, T] \times [0, 1]$. Then let

$$C(t, h)\ (x) = a(t, x)\ h(x),\ B(t, h)\ (x) = \gamma\ (t, x)\ \sqrt{1 + h^2(x)}$$

and reformulate (4.1) by the stochastic evolution equation on H :

$$z(t) = T(t)z_0 + \int_{t_0}^{t} T(t - s)\ C(s, z(s))\ ds$$

$$+ \int_{t_0}^{t} T(t - s)\ B(s, z(s))\ d\beta(s). \qquad \ldots(4.2)$$

By making formal substitutions

$$u(t) = C(t, z(t)),\ v(t) = B(t, z(t))$$

in (4.2), it is easy to observe that (4.2) is equivalent to the following pair of equations

$$u(t) = C\left(t,\ T(t)z_0 + \int_{t_0}^{t} T(t - s)\ u(s)\ ds + \int_{t_0}^{t} T(t - s)v(s)\ d\beta(s)\right) \qquad \ldots(4.3)$$

$$v(t) = B\left(t,\ T(t)z_0 + \int_{t_0}^{t} T(t - s)\ u(s)\ ds + \int_{t_0}^{t} T(t - s)v(s)\ d\beta(s)\right) \qquad \ldots(4.4)$$

for $t \in [0, T]$. Now the problem of solving (4.2) is equivalent to the problem of solving

the equations (4.3) – (4.4). It is easy to observe that the system (4.3) – (4.4) is a special case of the system (2.9) – (2.10) and $T(t)$, $C(t, z)$ and $B(t, z)$ satisfy the assumptions of Theorem 1. Hence, Theorem 1 yields the existence and uniqueness of the solution of the system (4.3) – (4.4) which in turn guarantee the existence and uniqueness of solution of the equation (4.2). Thus we get the existence and uniqueness of the mild solution of the equation (4.1).

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