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HAMILTONIAN DECOMPOSITIONS OF PRODUCTS OF CYCLES

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Whereas a Cartesian product of cycles is known to admit a Hamiltonian decomposition, we show that if a Kronecker product of cycles is connected, then it admits a Hamiltonian decomposition, otherwise all of its connected components are isomorphic to one another and each admits a Hamiltonian decomposition. We further derive a sufficient condition for an analogous decomposition of a strong product of cycles.

1. Introduction

We consider the Cartesian product, Kronecker product and strong product of graphs, and address the question of whether or not a product of cycles admits a 'Hamiltonian decomposition'. Whereas a cartesian product of (any number of) cycles is known to admit such a decompsotion[1,2], we present a characterization for a similar decomposition of a Kronecker product of cycles, and a sufficient condition for an analogous decomposition of a strong product of cycles. The major thrust of this paper is on decomposition of a Kronecker product of cycles.

By a graph we mean a finite, simple and undirected graph. Let $G = (V, E)$ and $H = (W, D)$ be graphs. The cartesian product (■ -product), Kronecker product (× -product) and strong product (■ -product) of G and H are respectively denoted by $G \square H$, $G \times H$, and $G \blacksquare H$, and are defined as follows :

$$V(G \square H) = V(G \times H) = V(G \blacksquare H) = V \times W$$

$$E(G \square H) = \{\{(u, x), (v, y)\} \mid \text{either } u = v \text{ and } \{x, y\} \in D \text{ or } x = y \text{ and } \{u, v\} \in E\}$$

$$E(G \times H) = \{\{(u, x), (v, y)\} \mid \{u, v\} \in E \text{ and } \{x, y\} \in D\}$$

$$E(G \blacksquare H) = E(G \square H) \cup E(G \times H).$$

Note that each of the three products is commutative and associative up to isomorphism. Also, $E(G \square H) \cap E(G \times H) = \phi$. We remark here that there is no unanimity on the terminology or notation concerning graph products. For example, Kronecker product is also known as direct product, categorical product or tensor product.

A graph is said to admit a cycle decomposition (resp. Hamiltonian decomposition) if its edge set can be partitioned into cycles (resp. Hamiltonian cycles). For example, a completge graph on an odd number of vertices admits a Hamiltonian decomposition[3]. For any undefined terms, we refer to Behzad[4].

Let C_n denote the cycle of length $n \geq 3$, where $V(C_n) = \{0, \ldots, n-1\}$, and for $n_1, \ldots, n_r \geq 3$, consider the graphs $C_{n_1} \square \cdots \square C_{n_r}$, $C_{n_1} \times \cdots \times C_{n_r}$ and $C_{n_1} \blacksquare \cdots \blacksquare C_{n_r}$. It is easy to see that the number of vertices in each of the three graphs is $n_1 \cdots n_r$ while the sizes are $r \cdot n_1 \cdots n_r$, 2^{r-1}, $n_1 \cdots n_r$ and $(r + 2^{r-1}) \cdot n_1 \cdots n_r$ respectively. Thus if $C_{n_1} \times \cdots \times C_{n_r}$ admits a Hamiltonian decomposition, then the number of cycles in such a decomposition is 2^{r-1}.

The following result includes a characterization for connectedness of the $\times$-product of two graphs.

Theorem 1.1[12] – Let G and H be nontrivial, connected graphs. If G and H are both bipartite, then the graph $G \times H$ consists of exactly two connected components, otherwise it is connected. ■

Corollary 1.2 – For $n_1, \ldots, n_r \geq 3$, the graph $C_{n_1} \times \cdots \times C_{n_r}$ is connected if and only if at most one n_i is even. ■

It is known that each of the $\square$-product and $\blacksquare$-product of graphs is connected if and only if the factor graphs are all connected[10].

The next lemma will be useful in the sequel.

Lemma 1.3 – (1) Let $G = (V_1 \cup V_2, E)$ and $H = (W_1 \cup W_2, D)$ be connected, bipartite graphs. Then the two connected components of the graph $G \times H$ are respectively induced by $(V_1 \times W_1) \cup (V_2 \times W_2)$ and $(V_1 \times W_2) \cup (V_2 \times W_1)$.

(2) If G is a connected, bipartite graph and $n \geq 4$ is an even integer, then the graph $G \times C_n$ consists of two isomorphic connected components. ■

The following result is due to Hedetniemi.

Lemma 1.4[6] – The $\times$-product of a bipartite graph and any other graph is necessarily bipartite. ■

The next theorem is interesting and will be useful in the sequel.

Theorem 1.5[3] – The $\times$-product of graphs is distributive with respect to the edge-disjoint union of graphs (a property not holding for the $\square$-product or $\blacksquare$-product). ■

Based on a result of Aubert and Schneider[2], Alspach, Bermond and Sotteau[1] obtained the following theorem, which superseded earlier works of Kotzig[8] and Foregger[5].

Theorem 1.6[1] – For $n_1, \ldots, n_r \geq 3$, the graph $C_{n_1} \square \cdots \square C_{n_r}$ admits a Hamiltonian decomposition. ■

Bermond[3] mistakenly states that the $\times$-product of any two cycles admits a Hamiltonian decomposition. However, $\times$-product of two even cycles is not even connected.

Our central result is that if a $\times$-product of (any number of) cycles is connected, then it admits a Hamiltonian decomposition, otherwise all of its connected components are isomorphic to one another and each admits a hamiltonian decomposition.

2. MAIN RESULTS

The first theorem below gives a cycle decomposition of the graph $G \times C_n$: it relies on Theorem 1.5.

Theorem 2.1 – Let $G = (V, F)$ be a graph without isolated vertices and let $n \geq 3$. If n is odd (resp. even) then $G \times C_n$ admits a decomposition into $|E|$ (resp. $2 \cdot |E|$) cycles, each of length $2 \cdot n$ (resp. n). ■

Corollary 2.2 – Let G be a graph without isolated vertices and let H be a graph which admits a decomposition into (not necessarily equal-length) cycles. Then the graph $G \times H$ admits a decomposition into (not necessarily equal-length) cycles. ■

We next construct a decomposition of the graph $C_m \times C_n$ into certain equal-length cycles.

Theorem 2.3 – Let $m, n \geq 3$, and let $p = gcd(m, n)$ and $q = lcm(m, n)$. Then the graph $C_m \times C_n$ admits a decomposition into $2 \cdot p$ cycles, each of length q.

PROOF : Let m, n, p and q be as in the statement of the theorem, and consider the graph $C_m \times C_n$. Note that $|E(C_m \times C_n)| = 2 \cdot m \cdot n = 2 \cdot p \cdot q$.

Consider the following p vertices of $C_m \times C_n$: $(0, 0), (0, 1), \ldots, (0, p-1)$. For each $(0, j)$ in this sequence, we trace exactly two edge-disjoint cycles, each of which includes $(0, j)$ and is of length q.

The two cycles containing $(0, j)$ are given by the following sequences : $x_0, \ldots, x_{q-1}$ and $y_0, \ldots, y_{q-1}$ where $x_k = (a, b)$ and $y_k = (a, c)$ with $a = k \bmod m$, $b = (j + k) \bmod n$ and $c = (j - k) \bmod n$. Intuitively the first (resp. second) sequence is constructed as follows : (i) start with the pair $(0, j)$, (ii) increment the first co-ordinate modulo m and increment (resp. decrement) the second co-ordinate modulo n, and (iii) continue this process until all vertices in the sequence are pairwise distinct. It is easy to see that the two sequences correspond to edge-disjoint cycles, each of length q.

To conclude the proof, note that the resulting $2 \cdot p$ cycles are mutually edge-disjoint. ■

Corollary 2.4 – If m and n are relatively prime, then the graph $C_m \times C_n$ admits a Hamiltonian decomposition. ■

From subsequent discussion, it will follow that the converse of Corollary 2.4 is false. In particular, the existence of a hamiltonian cycle in $G \times H$ does not imply hamiltonicity of G and H. A slightly weaker form of this corollary appears in Jha and Slutzki[7].

We next present two lemmas which lead to a characterization for Hamiltonian decomposition of $C_m \times C_n$.

Lemma 2.6 – If $m \geq 3$ and $n \geq 3$ are integers such that one is odd and the other is even, then the graph $C_m \times C_n$ admits a Hamiltonian decomposition.

PROOF : Let $m, n \geq 3$ and assume that m is odd and n is even. It suffices to trace two edge-disjoint Hamiltonian cycles in $C_m \times C_n$.

Consider the following two sequences of vertices : $x_0, \ldots, x_{mn-1}$ and $y_0, \ldots, y_{mn-1}$ where $x_{2mi+j} = (a, b)$ and $y_{2mi+j} = (a, c)$ with $a = j \bmod m$, $b = 2 \cdot i + (j \bmod 2)$ and $c = (-b) \bmod n$, $0 \leq i \leq (n/2) - 1$, $0 \leq j \leq 2 \cdot m - 1$. It is straightforward to check that these sequences constitute edge-disjoint Hamiltonian cycles of the graph $C_m \times C_n$. ■

Lemma 2.7 – If $m \geq 3$ and $n \geq 3$ are odd integers, then $C_m \times C_n$ admits a Hamiltonian decomposition.

PROOF : Let $m \geq 3$ and $n \geq 3$ be odd integers, and consider the following two sequences of vertices of the graph $C_m \times C_n : x_0, \ldots, x_{mn-1}$ and $y_0, \ldots, y_{mn-1}$ where $x_{mi+j} = (j, b)$ and $y_{mi+j} = (j, c)$, $0 \leq i \leq n - 1$, $0 \leq j \leq m - 1$ with $b = (i + (j \bmod 2)) \bmod n$ and $c = (-b) \bmod n$. It is straightforward to check that these sequences constitute edge-disjoint hamiltonian cycles of the graph $C_m \times C_n$. ■

Note : Results of Lemmas 2.6 and 2.7 appear in Mingkun[9] also, though with a different proof.

We next consider the case when m and n are even.

Lemma 2.8 – If $m \geq 4$ and $n \geq 4$ are even integers, then the two connected components of the graph $C_m \times C_n$ are isomorphic, and each admits a hamiltonian decomposition.

PROOF : Let $m \geq 4$ and $n \geq 4$ be even integers, and consider the graph $C_m \times C_n$. By Lemmas 1.3 and 1.4, this graph is bipartite and consists of two isomorphic components.

Note that there is a natural bipartition of the vertex set of (the bipartite graph) C_m into the following sets: $V_0 = \{0, 2, \ldots, m - 2\}$ and $V_1 = \{1, 3, \ldots, m - 1\}$. Let W_0 and W_1 correspond to the analogous bipartition of $V(C_n)$. We consider the (sub) graph induced by $(V_0 \times W_0) \cup (V_1 \times W_1)$ and trace two edge-disjoint hamiltonian cycles in it.

Consider the following sequences of vertices : $x_0, \ldots, x_{(mn/2)-1}$ and $y_0, \ldots, y_{(mn/2)-1}$ where $x_{(mi+j)} = (j, b)$ and $y_{mi+j} = (j, c)$ with $0 \leq i \leq (n/2) - 1$, $0 \leq j \leq m - 1$, and $b = 2 \cdot i + (j \bmod 2)$, $c = (-b) \bmod n$. It is straightforward to check that these two sequences constitute Hamiltonian decomposition of the connected component mentioned above. ■

Lemma 2.6, 2.7 and 2.8 yield the following result.

Theorem 2.9 – For $m.\ n \geq 3$, the graph $C_m \times C_n$ amits a Hamiltonian decomposition if and only if either m or n is odd. If m and n are both even, then $C_m \times C_n$ consists of two isomorphic connected components, each of which admits a Hamiltonian decomposition. ■

Note that Theorem 2.9 subsumes Corollary 2.4. We next proceed to obtain a generalization of Theorem 2.9.

Lemmas 2.10 – For $n_1,...,n_r \geq 3$, the graph $C_{n_1} \times \cdots \times C_{n_r}$ admits a Hamiltonian decomposition if and only if at most one n_i is even.

Proof : (By induction on r) For $r = 1$, the lemma is trivial while for $r = 2$, it follows from Theorem 2.9.

Let $r \geq 3$. The "only if" part follows from Corollary 1.2. For the "if" part, assume that the number of even integers among $n_1,...,n_r$ is at most one and let n_r be odd. The graph $C_{n_1} \times \cdots \times C_{n_{r-1}}$ admits a Hamiltonian decomposition. The claim then follows from Theorem 1.5, and Lemmas 2.6 and 2.7. ■

We next consider the case when at least two integers among $n_1,...,n_r$ are even. The result is a generalization of Lemma 2.8.

Lemma 2.11 – Let $n_1,...,n_r \geq 3$. If the number of even integers among $n_1,...,n_r$ is at least two, then the graph $C_{n_1} \times \cdots \times C_{n_r}$ consists of isomorphic connected components, each of which admits a Hamiltonian decomposition.

Proof : (by induction on r) The induction basis ($r = 2$) follows from Lemma 2.8. For the induction step, let $r \geq 3$ and let n_r be even. The graph $C_{n_1} \times \cdots \times C_{n_{r-1}}$ either admits a Hamiltonian decomposition or consists of isomorphic connected components, each of which admits a Hamiltonian decomposition. Note that $C_{n_1} \times \cdots \times C_{n_{r-1}}$ is bipartite. Since the number of vertices in a hamiltonian bipartite graph is necessarily even, it follows that every Hamiltonian cycle in each connected component of $C_{n_1} \times \cdots \times C_{n_{r-1}}$ is of even length. By Lemma 1.3(2), the $\times$-product of each connected component of $C_{n_1} \times \cdots \times C_{n_{r-1}}$ and C_{n_r} consists of two connected components isomorphic to each other. By transitivity of the "isomorphism" property of graphs, it is clear that all connected components of $C_{n_1} \times \cdots \times C_{n_r}$ are isomorphic. That each such component admits a Hamiltonian decomposition follows from Theorem 1.5 and Lemma 2.8. ■

Observe that if the number of even integers among $n_1,...,n_r$ is $k \geq 1$, then the number of connected components in the graph $C_{n_1} \times \cdots \times C_{n_r}$ is exactly 2^{k-1}.

Lemmas 2.10 and 2.11 along with Corollary 1.2 yield the following theorem which is the central result of this paper.

Theorem 2.12 – Let $n_1,...,n_r \geq 3$. If the graph $C_{n_1} \times \cdots \times C_{n_r}$ is connected then it admits a Hamiltonian decomposition, otherwise all of its connected components are isomorphic to one another and each admits a Hamiltonian decomposition. ■

Corollary 2.13 – If $G_1,...,G_r$ are Hamiltonian graphs such that $|V(G_i)|$ is even for at most one i, then the graph $G_1 \times \cdots \times G_r$ contains at least 2^{r-1} edge-disjoint Hamiltonian cycles. ■

Theorems 2.12 and 1.5, and a generalization of Lemma 1.3(2) yield the following result.

Corollary 2.14 – Let $G_1,...,G_r$ be graphs each of which admits a Hamiltonian decomposition, and let $|V(G_1)| = n_i$. If the number of even integers among

$n_1,\ldots,n_r$ is at most one, then the graph $G_1 \times \cdots \times G_r$ admits a Hamiltonian decomposition. If the number of even integers among $n_1,\ldots,n_r$ is at least two, and the corresponding graphs are bipartite, then $G_1 \times \cdots \times G_r$ consists of isomorphic connected components, each of which admits a Hamiltonian decomposition. ∎

We state a conjecture related to Corollary 2.14.

Conjecture 1 – Let G be a non-bipartite graph such that $|V(G)|$ is even and G admits a hamiltonian decomposition, and let $n \geq 4$ be an even integer. Then the graph $G \times C_n$ admits a Hamiltonian decomposition. ∎

If Conjecture 1 is true, then the result of Corollary 2.14 will be strengthened and the following general statement will hold true.

Conjecture 2 – Let $G_1,\ldots,G_r$ be graphs, each of which admits a Hamiltonian decomposition. If the graph $G_1 \times \cdots \times G_r$ is connected, then it admits a Hamiltonian decomposition, otherwise it consists of isomorphic connected components, each of which admits a Hamiltonian decomposition. ∎

We have found that Conjecture 1 is true in certain cases. For example, let $(K_6 - 3K_2)$ be the graph obtained from K_6 by deleting a perfect matching. This graph and C_4 appear in Fig. 1 while a Hamiltonian decomposition of the graph $(K_6 - 3K_2) \times C_4$ appears in Fig. 2.

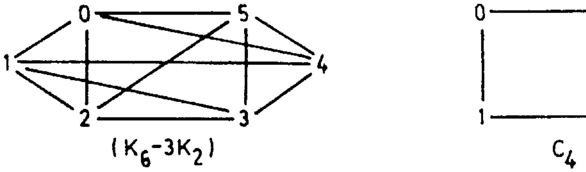

FIG. 1. Graphs $(K_6 - 3K_2)$ and C_4.

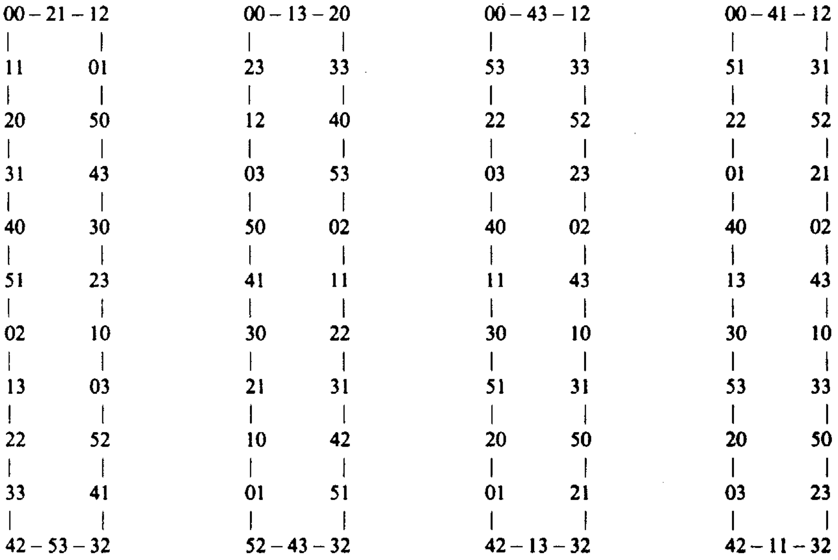

FIG. 2. Hamiltonian decomposition of $(K_6 - 3K_2) \times C_4$.

Finally we state a theorem relating to a $\blacksquare$-product of cycles. It is based on Theorems 1.6 and 2.12.

Theorem 2.15 – Let $n_1,\ldots,n_r \geq 3$. If at most one n_i is even, then the graph $C_{n_1} \blacksquare \cdots \blacksquare C_{n_r}$ admits a Hamiltonian decomposition. In general, $C_{n_1} \blacksquare \cdots \blacksquare C_{n_r}$ contains at least r edge-disjoint Hamiltonian cycles. $\blacksquare$

Mingkun[9] has shown that for any integers $m, n \geq 3$, the graph $C_m \blacksquare C_n$ admits a Hamiltonian decomposition. Whether or not this is true of a $\blacksquare$-product of $r \geq 3$ cycles remains an open problem.

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