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ON FUNCTIONAL RELATIONS – II

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A result concerning a double series, in the form of a series relation involving the generalized hypergeometric polynomial ${}_4F_3$ of unit argument and the digamma function, is established here with the help of generalized Riemann-Liouville operators of fractional calculus. As applications of the result established, a number of results, both known and new ones, can be derived.

1. INTRODUCTION AND DEFINITIONS

Recently Nishimato and Srivastava[2] gave certain series relations involving the generalized hypergeometric function ${}_pF_q$ and the Psi-function $\psi(x)$. Raina[3] provided further generalization of one of the results given by Nishimato and Srivastava[2]. The aim of this paper is to establish a result concerning a double series, in the form of a series relation involving the generalized hypergeometric polynomial ${}_4F_3$ of unit argument and the digamma function, and is established with the help of generalized Riemann-Liouville operators studied by Saigo[4]. One result, believed to be new, is also indicated as particular case of our main result.

A generalization of the familiar Riemann-Liouville fractional integral operator, was studied recently by Saigo[4] in the following form :

Let β and η be real. Then,

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)] = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \, {}_2F_1\left[\begin{matrix}\alpha+\beta, -\eta \\ \alpha\end{matrix}; 1-\frac{t}{x}\right] f(t)dt, \ \alpha>0 \qquad \ldots(1.1)$$

$$= \frac{d^n}{dx^n}\left\{ I_{0,x}^{\alpha+n,\ \beta-n,\ \eta-n}[f(x)]\right\}, \ \alpha \leq 0 \qquad \ldots(1.2)$$

where $0 < \alpha + n \leq 1$, n being a positive integer.

We have the following relation :

$$I_{0,x}^{\alpha,-\alpha,\eta}[f(x)] = R_{0,x}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt. \qquad \ldots(1.3)$$

The following result due to Saigo and Raina[5] is required here :

For $k > \max(0, \beta - \eta) - 1$,

$$I_{0,x}^{\alpha,\beta,\eta}\,[x^k] = \frac{\Gamma(k+1)\,\Gamma(-\beta+\eta+k+1)}{\Gamma(-\beta+k+1)\,\Gamma(\alpha+\eta+k+1)}\,x^{k-\beta}. \qquad \ldots(1.4)$$

2. The Theorem

Theorem 1 — Let, $P(x, y)$ denote the absolutely convergent double series given by

$$P(x, y) = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} A_{r,k}\,x^r\,y^k. \qquad \ldots(2.1)$$

Let

$$S_{\alpha_2;\lambda;n}^{\alpha_1,\beta_1;\eta_1}(x, y) = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} A_{r,k}\,{}_4F_3\left[\begin{matrix}-n, \beta_1+\lambda+r,\ \eta_1+\lambda+r, -\alpha_2+\lambda+k \\ \lambda+r,\ \alpha_1+\beta_1+\eta_1+\lambda+r,\ \lambda+k\end{matrix};\ 1\right] x^r\,y^k \qquad \ldots(2.2)$$

provided the double series on the right-hand side of (2.2) is absolutely convergent.

Then, for $\lambda > \max(0, -\alpha_1 - \beta_1 - \eta_1, \alpha_2, -\beta_1, -\eta_1)$, we have

$$\sum_{n=1}^{\infty}\frac{1}{n}\,S_{\alpha_2;\lambda;n}^{\alpha_1,\beta_1;\eta_1}(x, y) = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} A_{r,k}\,G_{\eta_1,\alpha_2}^{\alpha_1,\beta_1}(\lambda; r, k)\,x^r\,y^k \qquad \ldots(2.3)$$

where

$$G_{\eta_1,\alpha_2}^{\alpha_1,\beta_1}(\lambda; r; k) = [\psi(\lambda+r) + \psi(\alpha_1+\beta_1+\eta_1+\lambda+r) + \psi(\lambda+k) - \psi(\beta_1+\lambda+r) - \psi(\eta_1+\lambda+r) - \psi(-\alpha_2+\lambda+k)] \qquad \ldots(2.4)$$

provided that both sides of (2.3) are valid.

Outline of proof : To establish Theorem 1, we proceed on the lines as given in the paper by Raina[3]. We first evaluate

$$R_{0,y}^{-\alpha_2}\,[(x\,y)^{\lambda-1}\,P(x, y)]$$

operate on the result $I_{0,x}^{-\alpha_1,-\beta_1,\,\alpha_1+\eta_1}$, then differentiate the resulting equation with respect to λ to get the value of

$$I_{0,x}^{-\alpha_1,-\beta_1,\alpha_1+\eta_1}\,R_{0,y}^{-\alpha_2}\,[(xy)^{\lambda-1}\log(xy)\,P(x,y)]$$

in terms of Gamma-and Psi-functions. Next, we consider

$$R_{0,y}^{\alpha_2}\,I_{0,y}^{\alpha_1,\beta_1,\eta_1}\,[f(x, y)] = (xy)^{\lambda-1}\log(xy)\,P(x, y). \qquad \ldots(2.5)$$

On inverting eqn. (2.5) to obtain $f(x, y)$ and then substituting back the value of $f(x, y)$, thus obtained, in (2.5) and finally using the well-known Maclaurin's expansion for $\log xy$, the desired result in (2.3) is obtained with the help of (1.4) and certain known integrals.

On taking $A_{r,k} = 0$, $r = 0, 1, 2,\ldots;\ k = 1, 2,\ldots$ in Theorem 1, it reduces to the following :

Corollary 1 – Let $\Theta(x)$ denote the absolutely convergent series given by

$$\Theta(x) = \sum_{r=0}^{\infty} C_r x^r. \qquad \ldots(2.6)$$

Let

$$T^{\alpha_1,\beta_1;\eta_1}_{\alpha_2;\lambda;n}(x) = \sum_{r=0}^{\infty} C_r \, {}_4F_3\left[\begin{matrix}-n, \beta_1+\lambda+r, \eta_1+\lambda+r, -\alpha_2+\lambda \\ \lambda+r, \alpha_1+\beta_1+\eta_1+\lambda+r, \lambda\end{matrix}; 1\right] x^r \qquad \ldots(2.7)$$

provided the right-hand side of (2.7) exists.

Then, for $\lambda > \max(0, -\alpha_1 - \beta_1 - \eta_1, \alpha_2, -\beta_1, -\eta_1)$,

$$\sum_{n=1}^{\infty} \frac{1}{n} T^{\alpha_1,\beta_1;\eta_1}_{\alpha_2;\lambda;n}(x) = \sum_{r=0}^{\infty} C_r H^{\alpha_1,\beta_1}_{\eta_1,\alpha_2}(\lambda; r) x^r \qquad \ldots(2.8)$$

where

$$H^{\alpha_1,\beta_1}_{\eta_1,\alpha_2}(\lambda, r) = [\psi(\lambda+r) + \psi(\alpha_1+\beta_1+\eta_1+\lambda+r) + \psi(\lambda) - \psi(\beta_1+\lambda+r) - \psi(\eta_1+\lambda+r) - \psi(-\alpha_2+\lambda)] \qquad \ldots(2.9)$$

provided that both sides of (2.8) are valid.

3. Particular Cases

1. On taking $\alpha_2 = 0$ in Corollary 1, then, in view of Luke[1] [p. 104, eqn. 3.13, 3(10)], it immediately yields a result established earlier by Raina[3] [p. 173, eqn. (1.5)], which includes the result of Nishimato and Srivastava[2] [p. 105, eqn. (4.8)] as a particular case.

2. Again, if we take $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 0$ in the above corollary, we get, in view of the relation

$$\psi(z+1) - \psi(z) = \frac{1}{z}$$

the following :

$$\sum_{n=1}^{\infty} \sum_{r=0}^{\infty} \frac{1}{n} C_r \, {}_3F_2\left[\begin{matrix}-n, \eta_1+\lambda+r, \lambda-1 \\ \eta_1+\lambda+r+1, \lambda\end{matrix}; 1\right] x^r$$

$$= \sum_{r=0}^{\infty} C_r \left[\frac{1}{\eta_1+\lambda+r} + \frac{1}{\lambda-1}\right] x^r.$$

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