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OSCILLATING PERIODIC SOLUTIONS OF THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Sufficient conditions are presented for the existence of solutions to $x''' = p(t)g(x) + f(t)$ which are both oscillatory and period of period ω.

1. Introduction

We shall consider the nonlinear third order differential equation

$$x''' = p(t)\, g(x) + f(t) \qquad \ldots(1.1)$$

in which $p, f \in C([a, +\infty), R)$, $g \in C(R, R)$. We prove the existence of ω-periodic solutions for (1.1), under certain conditions. We shall first consider a periodic boundary value problem (PBVP for short) associated with (1.1) and obtain existence of solutions by employing the alternative method for nonlinear problems at resonance[2]. These solutions will then be related to the ω-periodic solutions of (1.1).

Equation (1.1) with $f(t) = 0$, was considered in Heidel[4], Nelson[6], Waltman[8], with a linear damping term and for the linear equation we mention the works of Lazer[5], Barret[1], and Swanson[7]. It seems that considerable work has not been done on nonhomogeneous third order nonlinear differential equations. In this paper, we wish to present the conditions under which (1.1) has solutions which are both oscillatory and periodic of period ω. In this study, we require no hypotheses regarding the uniqueness of solutions of initial value problems for (1.1). In section 2, we include the preliminaries on the alternative method for problems at resonance and mention an existence result for such a problem[3], which is critical in our study. Section 3 contains the main results.

2. Preliminaries

Let X be a real Hilbert space with inner product $(.,.)$, and norm $\|\cdot\|$. Let $L : D(L) \subset X \to X$ be a linear operator with finite dimensional kernel X_0 and range

$X_1 = X_0^{\perp}$ so that $X = X_0 \oplus X_1$. Let $P : X \to X$ be the projection operator with the range X_0 and let $H : X_1 \to D(L) \cap X_1$ be the partial inverse of L so that $H(I - P)\, L = LH(I - P) = I - P$. Since L has a nontrivial null space, this case is known as the 'problem at resonance'.

We are interested in solving the operator equation

$$Lx = Nx \quad x \in D\,(L) \qquad \ldots(2.1)$$

in which $N : X \to X$ is a nonlinear operator. Thus, $x = x_0 + x_1$, where $x_0 \in X_0$, $x_1 \in X_1$ is a solution of (2.1) if and only if[2].

$$x_1 = H(I - P)\, N(x_0 + x_1) \qquad \ldots(2.2)$$

$$PN\,(x_0 + x_1) = 0. \qquad \ldots(2.3)$$

Equation (2.2) and (2.3) are called the auxiliary and the bifurcation equations respectively.

The following result of Cesari and Kannan[3], will be used in our subsequent discussion.

Theorem 2.1 – Assume that H is compact and the following conditions hold :

(a) there exists a constant $J > 0$ such that $\|Nx\| \leq J$ for all $x \in X$;

(b) there exists a constant $R > 0$ such that for all $x = x_0 + x_1$ in X, $x_0 \in X_0$, $x_1 \in X_1$ with $\|x_0\| = R$ and $x_1 = H(I - P)\, N(x_0 + x_1)$, we have $(Nx, x_0) \leq 0$ (or ≥ 0).

Then eqn. (2.1) has at least one solution.

3. Main Result

We state our main result.

Theorem 3.1 – Suppose that p, f are periodic functions of period ω, and $M_f = \dfrac{1}{\omega} \displaystyle\int_0^{\omega} f(t)\, dt = 0$ and $g(0) = 0$. Assume that there exists an $\alpha > 0$ such that

$$\omega^3 \left[g_\alpha M_{|p|} + M_{|f|}\right] \leq \alpha \qquad \ldots(3.1)$$

where

$$g_\alpha = \mathrm{Max}_{|x| \leq \alpha} |g(x)|.$$

Then eqn. (1.1) has a solution x which is both oscillatory and periodic of period ω, provided any one of the following conditions hold :

$$\left.\begin{array}{ll} \text{(i)} & p \geq 0,\ x\, g(x) \geq 0 \\ \text{(ii)} & p \geq 0,\ x\, g(x) \leq 0 \\ \text{(iii)} & p \leq 0,\ x\, g(x) \geq 0 \\ \text{(iv)} & p \leq 0,\ x\, g(x) \leq 0 \end{array}\right\} \qquad \ldots(3.2)$$

In a natural way, we proceed as follows :

We consider the third order equation

$$x''' = p(t)\, g(x) + f(t) \qquad \ldots(3.3)$$

$$x(0) = x(\omega),\ x'(0) = x'(\omega),\ x''(0) = x''(\omega)$$

in which $p(t)$, $f(t)$ are continuous functions on $[0, \omega]$.

For $\alpha > 0$, define

$$\phi_\alpha(x) = \begin{cases} (\text{Sgn } x)\ \alpha & \text{if } |x| \geq \alpha \\ x & \text{if } |x| \leq \alpha. \end{cases} \qquad \ldots(3.4)$$

Now, consider the modified PBVP

$$x''' = p(t)\, g(\phi_\alpha(x)) + f(t) \qquad \ldots(3.5)$$

$$x(0) = x(\omega),\ x'(0) = x'(\omega),\ x''(0) = x''(\omega).$$

We shall prove the following result for the PBVP (3.5).

Lemma 3.2 – The PBVP (3.5) has atleast one solution provided there exists an $\alpha > 0$ such that, either

(1) $g(\alpha)\, M_p + M_f \geq 0 \geq g(-\alpha)\, M_p + M_f$

or

(2) $g(\alpha)\, M_p + M_f \leq 0 \leq g(-\alpha)\, M_p + M_f$.

Proof : We consider the space $X = L^2[0, \omega]$ with the inner product and norm. We define

$$Lx = x''' \qquad \ldots(3.6)$$

where $D(L) = \{x \in X : x, x' \text{ and } x'' \text{ are absolutely continuous, } x''' \in X,\ x(0) = x(\omega),\ x'(0) = x'(\omega),\ x''(0) = x''(\omega)\}$.

It is easy to see that X_0, the Kernel of L, consists of all constant functions, and the range L, X_1 is the class of all functions whose average is zero and $X = X_0 \oplus X_1$. We define the operators $P : X \to X$ and $H : X_1 \to D(L) \cap X_1$ as follows :

$$Px = \frac{1}{\omega} \int_0^\omega x(s)\, ds$$

and

$$Hx_1 = y_1$$

if and only if $y''' = x_1$, $y_1(0) = y(\omega)$, $y_1'(0) = y_1'(\omega)$, $y_1''(0) = y_1''(\omega)$ and $Py_1 = 0$.

Let $N_\alpha : X \to X$ be the nonlinear operator defined by

$$(N_\alpha(x))(t) = p(t)\, g(\phi_\alpha(x(t))) + f(t).$$

Clearly $D(N_\alpha) = X$ since ϕ_α is bounded. Thus the PBVP (3.5) may be translated into the operator equation

$$Lx = N_\alpha x. \qquad \ldots(3.7)$$

For $x \in X$, set $x = x_0 + x_1$, where $x_0 = Px$, and $x_1 = (I - P)\, x$.

Thus (3.7) is equivalent to the system of equations

$$x_1 = H(I - P)\, N_\alpha(x_0 + x_1) \qquad \ldots(3.8)$$

and

$$PN_\alpha\ (x_0 + x_1) = 0. \qquad \ldots(3.9)$$

Since ϕ_α is bounded, there exists a constant $J > 0$ such that $\|N_\alpha\, x\| \leq J$ for all $x \in X$. Also, given $x_0 \in X_0$, we can find a constant $C > 0$, independent of $x_0 \in X_0$, such that $\max_{t \in [0,\, \omega]} |x_1\ (t)| \leq C$, for any solution x_1 of (3.8).

Now choose $R > \alpha > 0$ large enough so that

$$|x_1\ (t)| \leq R - \alpha. \qquad \ldots(3.10)$$

Since $X_0 = \{1\}$, we see that if $x_0 \in X_0$, and $\|x_0\| = R$, then either $x_0 = R$ or $x_0 = -R$. If $x_0 = R$, then from (3.10) and under hypothesis (1)

$$\begin{aligned}(N_\alpha\ (x_0 + x_1),\, x_0) &= R \int_0^\omega [p(t)\ g(\phi_\alpha\ (R + x_1\ (t))) + f(t)]\ dt \\ &= R \int_0^\omega (p(t)\ g(\alpha) + f(t))\ dt \\ &= R\,\omega\ (g(\alpha)\ M_p + M_f) \geq 0.\end{aligned}$$

Similarly if $x_0 = -R$, we again see that $(N_\alpha\ (x_0 + x_1),\, x_0) \geq 0$. Thus the conditions of Theorem 2.1 are satisfied, and the proof is complete.

Lemma 3.3 – Assume that $M_f = \frac{1}{\omega} \int_0^\omega f(t)\ dt = 0$, $g(0) = 0$. Also, suppose that p and g satisfy any one of the conditions in (3.2).

Then the modified PBVP (3.5) has a solution x on $[0, \omega]$. Furthermore, for each solution of the PBVP (3.5), there exists a $\eta \in [0, \omega]$ such that $x(\eta) = 0$.

Proof : For any $\alpha > 0$, and for any one of the choices of p and g given in the above hypotheses, the hypotheses of Lemma 3.2 are satisfied. Therefore, the modified PBVP (3.5) has a solution x on $[0, \omega]$.

Now, we show the existence of some η in $[0, \omega]$ such that $x(\eta) = 0$. Integrating the differential equation in (3.5) on $[0, \omega]$, we get that

$$\int_0^\omega (p(t)\ g(\phi_\alpha\ (x(t)) + f(t))\ dt = 0. \qquad \ldots(3.11)$$

Suppose (3.2) (i) holds. If $x(t) \neq 0$ on $[0, \omega]$ then by continuity of x, either $x(t) > 0$, or $x(t) < 0$ for all $t \in [0, \omega]$. If $x(t) > 0$ for all $t \in [0, \omega]$, then $\phi_\alpha(x(t)) > 0$ and $g(\phi_\alpha(x(t))) \geq 0$ on $[0, \omega]$ and (3.11) cannot be satisfied.

By the same reasoning $x(t) \not\equiv 0$ for all t, and hence $x(\eta) = 0$, for some $\eta \in [0, \omega]$. Similar arguments complete the proof in all cases.

Remark 3.4 : The solution x found in Lemma 3.3 will have atleast two zeros on $[0, \omega]$, if $x(0) \neq 0$. From Lemma 3.3, it is clear that there exists $\eta \in [0, \omega]$ such that $x(\eta) = 0$. If $x(t)$ is of one sign, then (3.11) cannot be satisfied, and hence $x(t)$ has to change its sign atleast once on $[0, \omega]$. We see the existence of another zero on $[0, \omega]$ from the boundary conditions on x.

Lemma 3.5 – Assume that the hypotheses of Lemma 3.2 and Lemma 3.3 are satisfied. If there exists $\alpha > 0$ such that

$$\omega^3 [g_\alpha M_{|p|} + M_{|f|}] \leq \alpha \qquad \ldots(3.12)$$

then the PBVP (3.3) has a solution x on $[0, \omega]$.

Proof : From Lemma 3.3, it is clear that the modified PBVP (3.5) has a solution x on $[0, \omega]$, and that there exists an $\eta \in [0, \omega]$ such that $x(\eta) = 0$. From the boundary conditions $x(0) = x(\omega)$ and $x'(0) = x'(\omega)$ we see that there exists a $\xi \in [0, \omega]$ such that $x'(\xi) = 0$ and a $\mu \in [0, \omega]$ such that $x''(\mu) = 0$.

Since $|\phi_\alpha(x)| \leq \alpha$ for all $x \in (-\infty, \infty)$ we have from the differential equation in (3.5)

$$x''' \leq g_\alpha |p(t)| + |f(t)|. \qquad \ldots(3.13)$$

From the identity $x''(t) = x'(\mu) + \int_\mu^t x'''(s)\, ds$ and in view of (3.13), we have

$$|x''(t)| \leq g_\alpha \int_\mu^t |p(s)|\, ds + \int_\mu^t |f(s)|\, ds$$

and

$$|x''(t)| \leq g_\alpha \int_0^\omega |p(s)|\, ds + \int_0^\omega |f(s)|\, ds$$

$$= \omega [g_\alpha M_{|p|} + M_{|f|}].$$

Using the identity $x'(t) = x'(\xi) + \int_\xi^t x''(s)\, ds$

we have

$$|x'(t)| \leq \omega^2 [g_\alpha M_{|p|} + M_{|f|}]. \qquad \ldots(3.14)$$

Now from the identity $x(t) = \int_\eta^t x'(s)\, ds$ and (3.14), we have

$$|x(t)| \leq \omega^3 [g_\alpha M_{|p|} + M_{|f|}]$$

in view of (3.12).

Therefore, $x(t)$ is a solution of the PBVP (3.2).

Proof of Theorem 3.1

Consider the PBVP (3.2) and the modified PBVP (3.4). For any $\alpha > 0$, when p and g satisfy any one of the conditions in (3.2), there exists a solution x of (3.4) which has a zero on $[0, \omega]$.

Since the existence of a positive constant α such that (3.1) is satisfied is given, we see that the solution x obtained above for the modified PBVP (3.4) is a solution to the PBVP (3.2).

This solution x may be extended periodically on $[0, \infty)$ having atleast one zero on $[0, \omega]$, one zero on $[\omega, 2\omega]$, and so on, i.e., there exists arbitrarily large zeros of x. Also from Remark 3.3 it is clear that x changes its sign on $[0, \omega]$, $[\omega, 2\omega]$, and so on. Hence, the proof is complete.

The following example illustrates the theorem.

Example 3.6 –

$$x''' = \frac{1}{10^2\pi^3} x^3 - \left[\frac{1}{10^2\pi^3} \cos t + \frac{1}{10^8\pi^{12}} \sin^3 t\right]$$

$$p = \frac{1}{10^2\pi^3}, f = -\left[\frac{1}{10^2\pi^3} \cos t + \frac{1}{10^8\pi^{12}} \sin^3 t\right]$$

and $\omega = 2\pi$

Clearly the conditions of Theorem 3.5 are satisfied with $\alpha = 1$; hence there exists a solution x of the equation which is both oscillatory and periodic. Indeed $x(t) = \frac{1}{10^2 \pi^3} \sin t$ is such a solution.

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