Indian J. pure appl. Math., 23(11): 791-795, November 1992

ON *T*-TENSOR AND *v*-CURVATURE TENSOR OF *C*3-LIKE FINSLER SPACES

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(*Received* 20 *December* 1991; *accepted* 13 *July* 1992)

The purpose of the present paper is to study the properties of *T*-tensor and *V*-curvature tensor of *C*3-like Finsler spaces F^n of dimension $n \geq 4$ in which the Cartan's torsion tensor C_{ijk} is of the form $C_{ijk} = \mathcal{C}_{(ijk)} \{h_{ij} a_k + C_j b_k\}$. If C_{ijk} is of this form, the Finsler space F^n ($n \geq 4$) is said to be a *C*3-like Finsler space.

1. INTRODUCTION

There are three kinds of torsion tensor in Cartan's theory of Finsler space F^n. Two of them are (h) hv-torsion tensor C_{ijk} and (v)-torsion tensor P_{ijk} which are symmetric in all its indices. It is obvious that F^n is Riemannian if the tensor C_{ijk} vanishes and Deicke's theorem shows that F^n is also Riemannian even if the tensor C_i ($= g^{jk} C_{ijk}$) vanishes[1]. In 3-dimensional Finsler space F^3, C_{ijk} is always written in the form

$$LC_{ijk} = Hm_im_jm_k - J\,\mathcal{C}_{(ijk)}\{m_im_jn_k\} + I\,\mathcal{C}_{(ijk)}\{m_in_jn_k\} + Jn_in_jn_k \qquad ...(1.2)$$

where $\mathcal{C}_{(ijk)}\{\ \}$ denote the cyclic permutation of indices i, j, k and addifion H, I and J are main scalare and (l_i, m_i, n_i) is Moor's frame[2,4,8,9]. Here $l_i = \dot{\partial}_i L$ is the unit vector along the element of support, m_i is the unit vector along C_i i.e. $m_i = C_i/C$ where $C^2 = g^{ij} C_i C_j$ and n_i is a unit vector orthogonal to the vectors l_i and m_i. Since the angular metric tensor h_{ij} in F^3 can be written as

$$h_{ij} = m_im_j + n_in_j \qquad ...(1.3)$$

the eqn. (1.2) may be written as

$$C_{ijk} = \mathcal{C}_{(ijk)}\{h_{ij} a_k + C_iC_jb_k) \qquad ...(1.4)$$

where $$a_k = \frac{1}{L}\left(I\, m_k + \frac{J}{3}\, n_k\right)$$

and $$b_k = \frac{1}{LC^2}\left\{\left(\frac{H}{3} - I\right) m_k - \frac{4J}{3}\, n_k\right\}.$$

There are two approaches to introduce the *C*3-like Finsler space. One approach is analogous to the form (1.2) of C_{ijk} which is introduced by Numata[10]. A Finsler

space F^n ($n \geq 4$) is called a $C3$-like, if there exist four scalars H, I, J, K and two linearly independent vector fields X_i, Y_i which are all positively homogeneous of degree 0 (zero) in y^i and satisfy

$$LC_{ijk} = H X_i X_j X_k + I \mathfrak{C}_{(ijk)} \{X_i X_j Y_k\} + J \mathfrak{C}_{(ijk)} \{X_i Y_j Y_k\} + K Y_i Y_j Y_k. \quad ...(1.5)$$

The T-tensor and v-curvature tensors of the Finsler space satisfying (1.5) have been obtained by Numata[10]. The necessary and sufficient condition for a space to be a Berwald and Landsberg space have also been obtained in C3-like Finsler space.

The other approach is analogous to the form (1.4) of C_{ijk} which is introduced by Prasad and Singh[11]. A Finsler space F^n ($n \geq 4$) is called a C3-like Finsler space if there exist two vector fields a_k and b_k which are positively homogeneous of degree -1 and $+1$ respectively and satisfy

$$C_{ijk} = \mathfrak{C}_{(ijk)} \{h_{ij} a_k + C_i C_j b_k\}. \quad ...(1.6)$$

The purpose of the present paper is to find the T-tensor and v-curvature tensor of the Finsler space F^n satisfying (1.6).

2. The Properties of $C3$-Like Finsler Spaces

Since C_{ijk} is an indicatory tensor, it follows that a_k and b_k in (1.6) are indicatory tensor i.e. $a_0 = b_0 = 0$. Here 0 denote contraction with the element of support y^i. Contracting (1.6) with g^{jk}, we get

$$(1 - 2 C_b) C_i = (n + 1) a_i + C^2 b_i \quad ...(2.1)$$

where $C_b = C_i b^i$

This shows the following :

Proposition 2.1 – The three vectors a_i, b_i and C_i are linearly dependent vectors.

Contracting (2.1) with C^i, we get

$$(1 - 3 C_b) C^2 = (n + 1) C_a \quad ...(2.2)$$

where $C_a = C_i a^i$.

This result gives.

Proposition 2.2 – If C_i is perpendicular to b_i (i.e. $C_b = 0$), then $C_a = \dfrac{C^2}{n+1}$ and if C_i is perpendicular to a_i then $C_b = 1/3$.

Eliminating b_i from (2.1) and (1.6), we get

$$C_{ijk} = \mathfrak{C}_{(ijk)} \left\{ h_{ij} a_k - \frac{n+1}{C^2} C_i C_j a_k \right\} + \frac{3(1 - 2C_b)}{C^2} C_i C_j C_k. \quad ...(2.3)$$

If a_i is parallel to C_i i.e. $a_i = \dfrac{p}{n+1} C_i$ where p is some scalar then the above form of C_{ijk} reduces to

$$C_{ijk} = \frac{p}{n+1} (h_{ij} C_k + h_{jk} C_i + h_{ki} C_j) + \frac{q}{C^2} C_i C_j C_k \quad ...(2.4)$$

where $q = 3 (1 - p - 2C_b)$. ...(2.5)

Contracting (2.4) with g^{jk}, we get

$$C_i = (p + q) C_i.$$

Hence $p + q = 1.$ ...(2.6)

In view of (2.6) and (2.5), we get

$$p = 1 - 3C_b.$$

Thus (2.1) gives $b_i = \frac{C_b}{C^2} C_i.$

A Finsler space F^n is said to be a semi C-reducible Finsler space if C_{ijk} of F^n can be written in the form (2.4) (Matsumoto and others[5,7]).

The above discussions give the following :

Theorem 2.1 – In a C3-like Finsler space F^n, if a_i is parallel to C_i, then b_i is also parallel to C_i and F^n reduces to a semi C-reducible Finsler space.

3. The v-Curvature Tensor of C3-Like Finsler Spaces

The v-curvature tensor S_{hijk} and v-Ricci tensor S_{hj} of F^n are given by

$$S_{hijk} = C_{hkr} C^r_{ij} - C_{hjr} C^r_{ik} \quad ...(3.1)$$

$$S_{hj} = S_{hijk} g^{ik}. \quad ...(3.2)$$

In view of (3.1) and (1.6) the v-curvature tensor of $C3$-like Finsler space is given by

$$S_{hijk} = h_{hk} B_{ij} - h_{hj} B_{ik} + h_{ij} B_{hk} - h_{ik} B_{hj} + C^2 (C_i h_h - C_h b_i) (C_k b_j - C_j b_k) \quad ...(3.3)$$

where $B_{ij} = \frac{a^2}{2} h_{ij} + a_i a_j + C_i C_j a_r b^r + (C_i b_j + C_j b_i) C_a$

and $a^2 = g^{ij} a_i a_j.$

Contracting (3.5) with g^{ik}, we get

$$S_{hj} = -(n-3) B_{hj} - \alpha h_{hj} + C^4 b_h b_j + C^2 b^2 C_h C_j - C^2 C_b (b_h c_j + C_h b_j) \quad ...(3.4)$$

where $\alpha = \frac{a^2}{2} (n+1) + C^2 a_r b^r + 2C_a C_b$

and $b^2 = g^{ij} b_i b_j.$

Now we shall consider an $S3$-like and $C3$-like Finsler space. An n (≥ 4) dimensional Finsler space is called S3-like[3,6], if the v-curvature tensor S_{hijk} is of the form

$$S_{hijk} = \frac{S}{(n-1)(n-2)} (h_{hj} h_{ik} - h_{hk} h_{ij}) \quad ...(3.5)$$

where S is the v-scalar curvature defined by

$$S = S_{hijk} g^{hj} g^{ik}.$$

From (3.3) and (3.5) we get

$$\left(\frac{S}{n-1} - a^2\right)(h_{hj}\, h_{ik} - h_{hk}\, h_{ij}) = h_{hk}\, A_{ij} + h_{ij}\, A_{hk} - h_{hj}\, A_{ik} - h_{ik}\, A_{hj} + C^2\,(C_i b_h - C_h b_i)\,(C_k b_j - C_j b_k) \quad \ldots(3.6)$$

where $$A_{ij} = a_i a_j + C_i C_j a_r b^r + (C_i b_j + C_j b_i)\, C_a. \quad \ldots(3.7)$$

Now, if $a_i = \dfrac{p}{n+1}\, C_i$, then $b_i = \dfrac{C_b}{C^2}\, C_i$

and then $A_{ij} = \lambda\, C_i\, C_j$

where $$\lambda = \frac{p}{n+1}\left(\frac{p}{n+1} + 3\, C_b\right).$$

Thus (3.6) reduces to

$$\left(\frac{S}{n-1} - a^2\right)(h_{hj}\, h_{ik} - h_{hk}\, h_{ij}) = \lambda\,(h_{hk}\, C_i\, C_j + h_{ij}\, C_h\, C_k - h_{hj}\, C_i\, C_k - h_{ik}\, C_h\, C_j).$$

contracting this equation with g^{ik}, we get

$$\left[\left(\frac{S}{n-1} - a^2\right)(n-2) + \lambda\, C^2\right] h_{hj} = -\lambda\,(n-3)\, C_h\, C_j$$

since rank of h_{hj} is $n-1$ while the rank of $C_h\, C_j = 1$, therefore we have

$$\lambda = C \text{ and } a^2 = \frac{S}{n-1}.$$

Thus we have the following.

Theorem 3.1 — If $C3$-like Finsler space is S3-like, then we have the equation (3.6). Furthermore, if $a_i = \dfrac{p}{n+1}\, C_i$ then $\dfrac{p}{n+1} + 3\, C_b = 0$ and $a^2 = \dfrac{S}{n-1}$.

4. The T-Tensor of $C3$-like Finsler Space

The T-tensor T_{hijk} of a Finsler space F^n is defined as

$$T_{hijk} = L\, C_{hij}|_k + l_h\, C_{ijk} + l_i\, C_{hjk} + l_j\, C_{hki} + l_k\, C_{hij} \quad \ldots(4.1)$$

where $|_k$ denote the v-covariant derivative with respect to Cartan's connection. Since

$$h_{ij}|_k = -L^{-1}\,(h_{ik}\, l_j + h_{jk}\, l_i)$$

we have from (1.6)

$$LC_{hij}|_k = \mathfrak{C}_{(hij)}\,\{Lh_{hi}\, a_j|_k - h_{ik}\,(l_h\, a_j + l_j\, a_h) + LC_i|_k\,(C_j\, b_h + C_h\, b_j) + LC_i\, C_j\, b_h|_k\}.$$

From (4.11), (4.2) and (1.6), we have the following form of T_{hijk} of C3-like Finsler space.

$$\begin{aligned} T_{hijk} = \mathfrak{C}_{(hij)} \{ & (L\, a_j|_k + l_k\, a_j + l_j\, a_k)\, h_{hi} \\ & + (L\, b_j|_k + b_k\, l_j + b_j\, l_k)\, C_h\, C_i \\ & + L\, C_h|_k\, (b_i\, C_j + b_j\, C_i) \\ & + b_i\, (l_h\, C_j\, C_k + l_j\, C_h\, C_k) \}. \end{aligned}$$

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