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A CHARACTERIZATION OF BINARY EULERIAN MATROIDS

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Eulerian matroids have been discussed by Welsh[1]. In this paper we prove that a binary matroid is eulerian if and only if the number of independent sets is odd.

1. INTRODUCTION

A matroid $M = (S, \mathcal{F})$ consists of a finite set S and a collection $\mathcal{F}$ of subsets of S with the following properties :

(1) $\phi \in \mathcal{F}$.

(2) If $X \in \mathcal{F}$ and $Y \subseteq X$ then $Y \in \mathcal{F}$.

(3) If $X \in \mathcal{F}$ and $Y \in \mathcal{F}$ and $|X| > |Y|$ then there exists an element $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

Members of $\mathcal{F}$ are called the independent sets of M. A maximal independent set of M is a base of M. A subset of S not belonging to $\mathcal{F}$ is said to be dependent. A minimal dependent subset of S is called a circuit of M. We follow the notations and terminologies of Welsh[2] and Recski[3].

A matroid M is called eulerian if S is a union of disjoint circuits of M. Two matroids M_1 and M_2 on S_1 and S_2 respectively are said to be isomorphic if there exists a bijection $\phi : S_1 \rightarrow S_2$ which preserves independence. Let F be a field. We say that a matroid M on a set S is representable over F if there exists a vector space V over F, a subset T of V and a bijective map $\phi : S \rightarrow T$ such that under ϕ, M is isomorphic to the matroid M induced on T by linear independence in V. A matroid M on S is called binary if it is representable over the Galois field $GF(2)$. Here we give a characterization of binary eulerian matroids in terms of independent sets.

We need the following definitions and results.

If $M = (S, \mathcal{F})$ is a matroid and $x \in S$ then the deletion of x from M (or restriction of M to S-$\{x\}$) denoted by $M_{\setminus\{x\}}$ is a matroid (S-$\{x\}$, $\mathcal{F}'$) where a subset Y of S-$\{x\}$ is in $\mathcal{F}'$ if and only if and only if $Y \in \mathcal{F}$. If x is not a loop, then a contraction of x in M denoted by $M_{/\{x\}}$ is the matroid (S-$\{x\}$, $\mathcal{F}'$) where a subset Y of S-$\{x\}$ is in $\mathcal{F}'$ if and only if $Y \cup \{x\}$ was independent before contraction. Matroids so obtained from M are known as minors of M.

Lemma 1 (Welsh[2], p. 162) — Any minor of a binary matroid is binary.

Lemma 2 (Welsh[2], p. 167) — Let M be a binary matroid on a set S, let $x \in S$ and let C be a circuit of M with $x \in C$. Then C-$\{x\}$ is a circuit of $M_{/\{x\}}$. If $x \notin C$ then either C is a circuit of $M_{/\{x\}}$ or is the disjoint union of two circuits of $M_{/\{x\}}$.

2. Main Result

Theorem — A binary matroid M on a set S is eulerian if and only if the number of independent sets of M is odd.

The above does not hold good for non-binary matroids as shown by the following example.

Example — Let $U_{6,2}$ be a uniform matroid of rank 2 on a six element set. This is a non-binary eulerian matroid. Then the number of independent sets in $U_{6,2}$ is 22, an even number. On the other hand the number of independent sets in $U_{4,2}$ is 11, an odd number but $U_{4,2}$ is not eulerian.

In general let $U_{k,2}$ be the uniform matroid of rank 2 on a k-element set (this is nonbinary for $k > 3$). One can see that it is eulerian if and only if k is congruent to 0 (Mod 3) and the number of its independent sets is odd if and only if k is congruent to 0 or 3 (Mod 4). Since 3 and 4 are relatively prime numbers any combination of the two properties can arise.

Proof of the Theorem : Let $M = (S, \mathcal{F})$ be a binary matroid. Let γ_M be the number of independent sets of M and $\gamma_M(x)$ be the corresponding number of independent sets of M containing x. Since a loop is a dependent set of M, we can assume without loss of generality that M is loopless. We proceed by induction on $|S|$. If $|S| = 2$ and M is eulerian then a base of M consists of a singleton subset of S and hence trivially the number of independent sets of M is odd.

Let now $|S| > 2$ and x be an arbitrary element of S. Form the matroids $M_{/\{x\}}$ by contracting x in M and $M_{\setminus\{x\}}$ by deleting x from M. By Lemma 1 both $M_{/\{x\}}$ and $M_{\setminus\{x\}}$ are binary matroids.

The independent sets of M are divided into two classes. The first class consists of the independent sets that include x and the second class those not containing x. An independent set in the first class, say X corresponds to the independent set $X' = X - \{x\}$ of $M_{/\{x\}}$ and an independent set X' of $M_{/\{x\}}$ corresponds to the independent set $X = X' \cup \{x\}$ of M in the first class. Also we note the one-one correspondence between independent sets of M containing x and independent sets of $M_{/\{x\}}$. So by induction,

$$\gamma = \gamma_{M_{/\{x\}}} = \gamma_M(x) \quad \text{... (1)}$$

with $\gamma \equiv 1$ (Mod 2) if and only if $M_{/\{x\}}$ is eulerian ... (2)

Further, if Y is an independent set of M in the second class, then Y is an independent set of $M_{\setminus\{x\}}$. So, we conclude from the above that

$$\gamma_M = \gamma_{M_{/\{x\}}} + \gamma_{M_{\setminus\{x\}}}. \quad \text{... (3)}$$

Therefore if M is eulerian then by Lemma 2, $M_{/\{x\}}$ is eulerian. However $M_{\setminus\{x\}}$ is not eulerian. By induction and by (2) and (3)

$$\gamma_M \equiv 1 + 0 = 1 \text{ (Mod 2)};$$

i.e. $\gamma_M \equiv 1$ (Mod 2).

This proves the only if part.

For the if part assume that $M = (S, \mathcal{F})$ is not an eulerian matroid. If for some $x \in S$ neither of $M_{/\{x\}}$ and $M_{\setminus\{x\}}$ is eulerian then by induction and by (2) and (3) we have

$$\gamma_M \equiv 0 + 0 = 0 \text{ (Mod 2)}.$$

If however, $M_{\setminus\{x\}}$ is eulerian then by Lemma 2, $M_{/\{x\}}$ is eulerian. Similarly, if $M_{/\{x\}}$ is eulerian then $M_{\setminus\{x\}}$ is eulerian. Thus, if one of $M_{/\{x\}}$ and $M_{\setminus\{x\}}$ is eulerian then both are eulerian.

Consequently,

$$\gamma_M \equiv 1 + 1 = 0 \text{ (Mod 2)}$$

i.e. $\gamma_M \equiv 0$ (Mod 2) if M is not eulerian.

This completes the proof of the theorem.

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References

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