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EXISTENCE OF KY FAN'S BEST APPROXIMANT FOR SET-VALUED MAPS

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We prove the existence of Ky Fan's best approximant for set valued maps with open inverse values.

Ky Fan's best approximation theorem states that if K is a nonempty compact convex subset of a locally convex Hausdroff topological vector space E and $f : K \to E$ is continuous, then f has a fixed point or there exist a point $x \in K$ and a continuous seminorm p on E such that $p(x - f(x)) = \inf \{p(f(x) - z) : z \in K\}$.

This theorem has been generalized to set-valued map by several authors, for example, see Ha[2] and Sehgal and Singh[6]. In this note, we shall prove the existence of best approximant for a set valued map with open inverse values. Our main tools are the following results due to Horvath[3] and Lassonde[4].

Throughout let E denote a locally convex Hausdorff topological vector space, p a continuous seminorm on E and 2^E the set of all nonempty subsets of E. If $K \subset E$ and $y \in E$, define $Q_p^K(y) = \{x \in K : p(x-y) = \inf\{p(z - y) : z \in K\}\}$. It is well-known that[5] when K is a nonempty compact convex subset of E, then the map $Q_p^K : E \to K$ is an upper-semicontinuous set-valued map with nonempty compact convex values.

If $N \in \mathbb{N}$, let $\langle N \rangle$ be the set of all nonempty subsets of $\{0, 1, 2, ..., N\}$, $\Delta_N = co\ \{e_0, e_1, ..., e_N\}$ be the standard simplex of dimension N, where $\{e_0, e_1, ..., e_N\}$ is the canonical basis of $\mathbb{R}^{N+1}$ and for $J \in \langle N \rangle$, let $\Delta_J = co\ \{e_j : j \in J\}$.

Lemma 1 (Horvath[3]) — Let X be a topological space and $F : \langle N \rangle \to X$. Suppose for each $J \in \langle N \rangle$, let $F(J)$ be a nonempty contractible subset of X and for all $J, J' \in \langle N \rangle$ such that $J \subseteq J'$, $F(J) \subseteq F(J')$. Then there exists a continuous function $f : \Delta_N \to X$ such that $f(\Delta_J) \subset F(J)$ for all $J \in \langle N \rangle$.

Also we need the following result due to Lassonde[4] which is a generalization of Kakutani fixed point theorem. Let us prefer to state it in the following form.

Lemma 2 (Lassonde[4]) — Let $F : \Delta_N \to \Delta_N$ be a set valued map such that $F = F_n \circ F_{n-1} \circ \ldots F_1 \circ F_0$ that is, $X_0 = \Delta_N \xrightarrow{F_0} X_1 \xrightarrow{F_1} X_2 \xrightarrow{F_2} \ldots \xrightarrow{F_n} X_{n+1} = \Delta_N$ where each F_i is either single-valued continuous function (in which case X_{i+1} is assumed to be Hausdorff topological space) or upper semicontinuous set-valued function with $F_i(x)$ a nonempty compact convex subset of X_{i+1} (in which case X_{i+1} is assumed to be a convex subset of a Hausdorff topological vector space). Then there exists a point $x_0 \in \Delta_N$ such that $x_0 \in F(x_0)$.

We now prove our theorem. We follow the method of Tarafdar and Yuan[5].

Theorem 1 — Let K be a nonempty compact convex subset of E. Let $S : K \to 2^E$ be a set-valued map such that

(i) $S^{-1}(x)$ is open for all $x \in E$

(ii) for each open set $F \subset K$, the set $\bigcap_{y \in F} S(y)$ is empty or contractible.

(iii) $S(K)$ is contractible.

Then there exist $x_0 \in K$ and $y_0 \in S(x_0)$ such that

$$p(x_0 - y_0) = \inf \{p(y_0 - z) : z \in K\}.$$

PROOF : Since K is compact, there exists a finite subset $\{x_0, x_1, .., x_N\}$ of $S(K)$ such that $K = \bigcup_{i=0}^{N} S^{-1}(x_i)$. Define $F : \langle N \rangle \to S(K)$ as follows :

$$F(J) = \begin{cases} \bigcap \left\{ S(y) : y \in \bigcap_{j \in J} S^{-1}(x_j) \right\} & \text{if } \bigcap_{j \in J} S^{-1}(x_j) \neq \phi \\ S(K) & \text{otherwise.} \end{cases}$$

It is clear that if $y \in \bigcap_{j \in J} S^{-1}(x_j)$, then $x_j \in S(y)$ for all $j \in J$. Thus $F(J)$ is nonempty contractible. Further $F(J) \subseteq F(J')$ whenever $J \subseteq J'$. By the Lemma, there exists a continuous function $f : \Delta_N \to S(K)$ such that $f(\Delta_J) \subset F(J)$ for all $J \in \langle N \rangle$. Let $\{g_i : i \in \{0, 1, ..., N\}$ be a continuous partition of unity subordinated to the covering $\{S^{-1}(x_i) : i \in \{0, 1, ..., N\}\}$, that is, for each i, $g_i : K \to [0, 1]$ is continuous, $\{y \in K : g_i(y) \neq \phi\} \subset S^{-1}(x_i)$ and $\Sigma_{i=0}^{N} g_i(y) = 1$ for all $y \in K$. Now define $g : K \to \Delta_N$ by $g(y) = (g_0(y), g_1(y), ..., g_N(y))$ for all $y \in K$. Then g is continuous. Further, $g(y) \in \Delta_{J(y)}$ for all $y \in K$ where $J(y) = \{i : g_i(y) \neq 0\}$. Therefore, $f \circ g(y) \in f(\Delta_J(y)) \subset F_{J(y)} \subset S(y)$. Consider $G := g \circ Q_p^K \circ f : \Delta_N \to \Delta_N$. Now by Lemma 2, there exists $z_0 \in \Delta_N$ such that $z_0 \in G(z_0)$. Let $y_0 = f(z_0)$. Then $y_0 \in f \circ g \circ Q_p^K \circ f(z_0)$, that is, there exists $x_0 \in Q_p^K(y_0)$ so that $y_0 \in f \circ g(x_0) \in S(x_0)$. This completes the proof.

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