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THE COMMON MINIMAL DOMINATING GRAPH

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For any graph G, its common minimal dominating graph $CD(G)$ is the graph having the same vertex set as G with two vertices in $CD(G)$ adjacent if and only if there exists a minimal dominating set in G containing them. In this paper, characterizations are given for graphs G for which $CD(G)$ is connected and $CD(G) = G$. Further, several new results are developed relating to this new graph.

1. INTRODUCTION

The graphs considered here are finite, undirected without loops or multiple edges. We denote by p the order (i.e., number of vertices) of such a graph G. Any undefined term in this paper may be found in Harary[2].

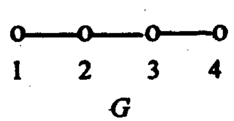
Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D of G is minimal if for any vertex $v \in D$, $D - \{v\}$ is not a dominating set of G. The minimum and maximum cardinalities taken over minimal dominating sets of G are the domination number $\gamma(G)$ and upper domination number $\Gamma(G)$ of G respectively.

The neighbourhood graph $N(G)$ of a graph G is the graph having the same vertex set as G with two vertices adjacent in $N(G)$ if and only if they have a common neighbour in G (see Brigham and Dutton[1]).

Now we define a similar type of a graph namely "the common minimal dominating graph" as follows :

The common minimal dominating graph $CD(G)$ of a graph G is the graph having the same vertex set as G with two vertices adjacent in $CD(G)$ if and only if there exists a minimal dominating set in G containing them.

In Fig. 1, a graph G and its common minimal dominating graph $CD(G)$ are shown.



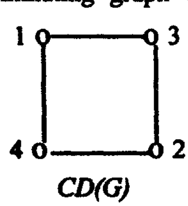


FIG. 1.

2. Results

Let $\overline{G}$ denote the complement of G.

Theorem 1 — For any graph G,

$$\overline{G} \subseteq CD(G) \quad \ldots (1)$$

and $\overline{G} = CD(G)$ if and only if every minimal dominating set of G is independent.

PROOF : If $(u, v) \in E(\overline{G})$, then extend $\{u, v\}$ to a maximal independent set S of vertices in G. Since S is also a minimal dominating set of G, we obtain $\overline{G} \subseteq CD(G)$.

Now we prove the second part.

If every minimal dominating set of G is independent, then two vertices adjacent in G cannot be adjacent in $CD(G)$. Thus $CD(G) \subseteq \overline{G}$, and together with (1) we see that $CD(G) = \overline{G}$.

Conversely, $CD(G) \subseteq \overline{G}$ implies that two vertices in the same minimal dominating set S are not adjacent in G, i.e., S is independent.

Let $\Delta(G)$ denote the maximum degree of G.

Theorem 2 — For any graph G with p vertices, $p \geq 2$, $CD(G)$ is connected if and only if $\Delta(G) < p - 1$.

PROOF : Let $\Delta(G) < p - 1$ and u, v be any two vertices of G. If $(u, v) \notin E(G)$, then by Theorem 1, $(u, v) \in CD(G)$. If $(u, v) \in E(G)$ and some vertex w distinct from both u and v is adjacent to neither, then again by Theorem 1, in $CD(G)$ the path uwv joins u to v. It only remains to consider $(u, v) \in E(G)$ and every other vertex w is adjacent to at least one of u and v. Then $\{u, v\}$ is a minimal dominating set of G and hence $(u, v) \in CD(G)$. Thus $CD(G)$ is connected.

Conversely, suppose $CD(G)$ is connected. If possible, suppose $\Delta(G) = p - 1$ and u is a vertex of degree $p - 1$. Then u is an isolated vertex in $CD(G)$. Since G has at least two vertices, $CD(G)$ has at least two components, a contradiction. Thus $\Delta(G) < p - 1$.

Theorem 3 — For any graph G,

$$\gamma(CD(G)) = p \quad \ldots (2)$$

if and only if $G = K_p$.

PROOF : Suppose (2) holds. Then $CD(G) = \overline{K}_p$ and hence by (1), $\overline{G} = \overline{K}_p$. Thus, $G = K_p$.

Conversely, suppose $G = K_p$. Then every minimal dominating set of G is independent. Hence by Theorem 1, $CD(G) = \overline{G} = \overline{K}_p$.

Thus (2) holds.

In a graph G, a vertex and an edge incident with it are said to cover each other. A set of vertices which covers all the edges is a vertex cover of G. The vertex covering number $\alpha_0(G)$ of G is the minimum number of vertices in a vertex cover. A set S of vertices in G is independent if no two vertices in S are adjacent. The independence number $\beta_0(G)$ of G is the maximum cardinality of an independent set of vertices.

Now we obtain some upper bounds for $\gamma(CD(G))$.

Let $\omega(G)$ denote the clique number of G.

Theorem 4 — For any graph G,

$$\gamma(CD(G)) \leq \omega(G). \qquad \ldots (3)$$

PROOF : By Theorem 1,

$$\begin{aligned}\gamma(CD(G)) &\leq \gamma(\overline{G}) \\ &\leq \beta_0(\overline{G}) \\ &\leq \omega(G).\end{aligned}$$

Theorem 5 — For any graph G,

$$\gamma(CD(G)) \leq p - \Gamma(G) + 1. \qquad \ldots (4)$$

PROOF : Let S be a minimal dominating set of G with $|S| = \Gamma(G)$. Then $\omega(CD(G)) \geq \Gamma(G)$. Thus (4) follows from the facts that $\gamma(G) \leq p - \Delta(G)$ and $\Delta(G) \geq \omega(G) - 1$ for any graph G.

Let $\alpha_0(G)$ denote the vertex covering number of G, i.e., the minimum number of vertices in a vertex cover and use the fact that $\alpha_0(G) + \beta_0(G) = p$.

Corollary 5.1 — For any graph G,

$$\gamma(CD(G)) \leq \alpha_0(G) + 1. \qquad \ldots (5)$$

Let $\delta(G)$ denote the minimum degree of G.

Theorem 6 — For any graph G,

$$\gamma(CD(G)) \leq 1 + \delta(G). \qquad \ldots (6)$$

PROOF : By (1), $\Delta(CD(G)) \geq \Delta(\overline{G}) = p - 1 - \delta(G)$. Hence (6) follows from the first fact used in the proof of Theorem 5.

To prove our next two results we make use of the following results from Harary[2].

Theorem A — A graph G is eulerian if and only if every vertex of G is of even degree.

Theorem B — If for all vertices v of G, *deg* $(v) \geq p/2$ where $p \geq 3$, then G is hamiltonian.

A graph G is odd if and only if every vertex of G is of odd degree.

Let *diam*(G) denote the diameter of G.

Theorem 7 — If G is an odd graph with $\Gamma(G) = diam(\overline{G}) = 2$, then $CD(G)$ is eulerian.

PROOF : Since p is even, every vertex in G is nonadjacent to an even number of vertices. Hence every vertex in $\overline{G}$ is of even degree and by Theorem A, $\overline{G}$ is eulerian.

Since for any two adjacent vertices u, v in G, there exists a vertex w which is not adjacent to both u and v and further there is no minimal dominating set in G containing u and v, by Theorem 1, $CD(G) = \overline{G}$.

Hence $CD(G)$ is eulerian.

Let $\lfloor x \rfloor$ denote the greatest integer not greater than x.

Theorem 8 — Let G be a graph of order at least three satisfying one of the following conditions :

(i) $\Delta(G) < \lfloor p/2 \rfloor$;

(ii) $\Delta(G) = \lfloor p/2 \rfloor$ and for every vertex v of G with $\deg(v) = \lfloor p/2 \rfloor$, there exists a vertex $u \in N(v)$ such that u is adjacent to every vertex in $V - N(v)$.

Then $CD(G)$ is hamiltonian.

PROOF : Suppose (i) holds. Then $\delta(\overline{G}) \geq p/2$ and hence by (1) and Theorem B, $CD(G)$ is hamiltonian.

Suppose (ii) holds. Then $\{u, v\}$ is a dominating set of G and further it is minimal, since there exist two vertices $u_1 \in N(u) - N(v)$ and $v_1 \in N(v) - N(u)$. Hence by (1), *deg* (v) in $CD(G) \geq p/2$.

Also by (i), for any vertex u with $deg(u) < \lfloor p/2 \rfloor$, $deg(u)$ in $CD(G) \geq p/2$.

Hence by Theorem B, $CD(G)$ is hamiltonian.

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