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MATRIX MAPS ON SEQUENCE SPACES ASSOCIATED WITH SETS OF INTEGERS

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Generalizing the space of statistically convergent sequence the space c_Φ is introduced, where Φ is a nonempty family of subsets of natural numbers. The matrix classes $(l_\infty \cap c_\Phi, c)$ and $(c_\Phi \cap l_\infty, m(\phi))$ are characterized which include a number of known results. Further, the classes $(w(p), X)$, $(l(p), X)$ and (c, X) are characterized for any closed subspace X of l_∞.

§1. The prime object of this note is to study matrix transformations on sequence spaces which generalize the space $\bar{c}$ of all statistically convergent sequences. The latter depends on the concept of density of subsets of the set N of natural numbers. In fact, a subset A of N is said to have density $\delta(A)$ if $\delta(A) = \lim_n n^{-1} \sum_{k=1}^{n} \chi_A(k)$ exists, where χ_A is the characteristic function of A. A complex sequence $x = (x_k)_{k \in N}$ (or simply (x_k)) is said to be statistically convergent to a, that is, stat-lim $x = a$ or $x_k \xrightarrow{\text{stat}} a$ if (Fast[3]) for every $\varepsilon > 0$.

$$\delta[\{k \in N : |x_k - a| \geq \varepsilon\}] = 0.$$

The number a, necessarily unique, is called the statistical limit of x. A significant fact concerning bounded statistically convergent sequences is the decomposition theorem[2,3] which states that $x \in \bar{c} \cap l_\infty$ iff x can be written as $y + z$, $y \in c$, $z \in \delta_0 \cap l_\infty$, where c is the space of all convergent sequences and δ_0 consists of all sequences z such that $\delta[\{k \in N : z_k \neq 0\}] = \delta\ (\text{supp } z) = 0$.

The class of all sets $A \subset N$ with $\delta(A) = 0$ is a special case of a full class which has applications in the study of matrix operators. A class Φ of subsets of N is said to be full if (Sember and Freedman[15])

(a) $U\ \{A : A \in \Phi\} = N$ (covering property),

(b) if $B \subset A$ and $A \in \Phi$, then $B \in \Phi$ (hereditary property), and

(c) if $t = (t_k)$ is a real sequence such that

$$\sum_{k \in S} |t_k| < \infty \text{ for each } S \in \Phi, \text{ then } \sum_k |t_k| = \sum_{k=1}^{\infty} |t_k| < \infty.$$

In other words, a covering hereditary class Φ is full if $(\chi_\Phi)^\alpha = l_1$, where E^α denotes the α-dual of any sequence space E (Köthe[5]), l_1 is the space of absolutely convergent sequences and $\chi_\Phi = \{\chi_s : S \in \Phi\}$.

For any nonempty class Φ of subsets of N, we write $c_\Phi = c + \delta_\Phi$, where $\delta_\Phi = \{x : \text{Supp } x \in \Phi\}$. The space c, $\bar{c}$ and the space l_∞ of all bounded sequences are instances of c_Φ for suitable choices of Φ. It is easy to see that $\chi_\Phi \subset \delta_\Phi \subset c_\Phi$ and Φ is full iff $(\delta_\Phi \bigcap l_\infty)^\alpha = l_1$. If $x = y + z \in c_\Phi$, $z \in \delta_\Phi$ and $\lim y = a$, we call a, a c_Φ-limit of x. The uniqueness of c_Φ-limit, the linearity of c_Φ, the completeness of $c_\Phi \bigcap l_\infty$ and such properties are not of immediate relevance. These have been discussed elsewhere (Rath[10]). It is also possible to extend the results of Rath and Tripathy[12] in the above light.

The spaces c_Φ are significant for the study of matrix transformations. Sember and Freedman[15] characterized the matrix class (χ_Φ, c) when Φ is full and deduced a classical result due to Hahn[4]. In the present paper, characterizations of certain matrix classes involving the space c_Φ are studied in Theorems 1 and 2, which reveal more immediate connections between such matrix classes and full classes. In Theorems 3, 4, 5, matrix classes involving arbitrary closed linear subspaces of l_∞ are studied which include several known cases.

As per standard notations, we write (X, Y) for the class of all matrices which map X into Y. The spaces c_0, l_p, $l(p)$, w_p, $w(p)$, c have their usual meanings[7]. Besides, the space $m(\Phi)$ (Sargent[14]) finds use in these discussions. We recall that for any positive nondecreasing sequence $\phi = (\phi_n)$ such that $\left(\frac{\phi_n}{n}\right)$ is nonincreasing, $m(\phi)$ consists of all sequences x such that $\sup_s \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty$ (or equivalently, $\sup_s \left| \frac{1}{\phi_s} \sum_{k \in \sigma} x_k \right| < \infty$ by Lemma 1 of Rath and Tripathy[11] where $s \in N$ and $\sigma \in C_s$, the class of all subsets of N having at the most s elements. It is known[14] that $m(\phi) = l_1$ or l_∞ according as $\phi_s = 1$ or $\phi_s = s$ for all $s \in N$.

2. The Following Lemmas are Crucial in the Proofs of the Theorems

Lemma 1 (Hahn[4]) — A matrix $A = (a_{nk}) \in (l_\infty, c)$ iff A maps all sequences of 0's and 1's into c.

Lemma 2 (Sember and Freedman[15], Proposition 1) — A covering hereditary class Φ of subsets of N is full iff any one of the following holds :

(a) If $t = (t_k)$ is such that $\sum_{k \in s} |t_k| < \infty$ for each $S \in \Phi$ then $t \in l_1$.

(b) If $\sup_n \sum_{k \in S} |a_{nk}| < \infty$ for each $S \in \Phi$, then

$$\sup_n \sum_k |a_{nk}| < \infty \qquad \text{... (2.1)}$$

(c) If $\sup_n \left| \sum_{k \in S} a_{nk} \right| < \infty$ for each $S \in \Phi$, then (2.1) holds.

Lemma 3 — A matrix $A = (a_{nk}) \in (l_\infty, m(\phi))$ iff

$$\sup_{s \in N, \sigma \in C_s} \sum_k \left| \frac{1}{\phi_s} \sum_{n \in \sigma} a_{nk} \right| < \infty. \qquad \text{... (2.2)}$$

This is a consequence of Theorem 1 of Rath and Tripathy[11].

Lemma 4 — For any linear subspace X of l_∞, the following are equivalent

(a) X is complete

(b) If $\sum_k a_{nk}$ converges uniformly to a_n for each $n \in N$ and for each $k \in N$, $a^k = (a_{nk})_{n \in N} \in X$, then $a = (a_n) \in X$.

PROOF : (a) $\Rightarrow$ (b) : Let $(a_{nk}), a^k$ be as in (b). Then $s^m = \sum_{k=1}^{m} a^k \in X$, $m \in N$ and $\| s^m - a \| = \sup_n \left| \sum_{k=m+1}^{\infty} a_{nk} \right| \to 0$ as $m \to \infty$, so that $a = (a_n) \in X$ whenever (a) holds.

(b) $\Rightarrow$ (a) : Let (x^m) be a Cauchy sequence in X and write $x^m = (x_k^m)_{k \in N}$. Then $x_k = \lim_m x_k^m$ exists for each $k \in N$ and $\|x^m - x\| \to 0$ as $m \to \infty$, where $x = (x_k) \in l_\infty$. If we write $a_{mk} = x_k^m - x_k^{m-1}$ ($x_k^0 = 0$ by convention), then $\sum_m a_{mk}$ converges uniformly to x_k, so that by (b) $x = (x_m) \in X$ which proves (a).

Remark 1 : If X is a closed linear subspace of l_∞ and f is a bounded linear functional on X, then $f(a) = \sum_k f(a^k)$, where a, a^k are as in Lemma 4. In particular, if $X = \bar{c} \cap l_\infty$ (which is complete[13]) and $f(x) =$ stat–lim x, $x \in \bar{c} \cap l_\infty$, then stat-lim $a = \sum_k$ stat-lim a^k, where a, a^k are as above. Similarly, if $X = c$ and $f(x)$ is the number to which $x \in \hat{c}$ is almost convergent, then a is almost convergent to $\sum \alpha_k$, whenever a, a^k are as above and a^k is almost convergent to α_k.

Lemma 5 (Lascarides and Maddox[6], Theorem 1) — The matrix $A = (a_{nk}) \in (l(p), l_\infty)$ iff (i) when $1 < p_k \le H$ for all k, there is $B > 1$ such that

$$M = \sup_n \sum_k |a_{nk}|^{q_k} B^{-q_k} < \infty, \quad \frac{1}{p_k} + \frac{1}{q_k} = 1 \text{ for all } k. \qquad \ldots (2.3)$$

and

(ii) when $0 < p_k \le 1$ for all k, we have

$$K = \sup_{n,k} |a_{nk}|^{p_k} < \infty. \qquad \ldots (2.4)$$

Lemma 6 (Maddox[8], Proof of Theorem 2) — If $1 < p_k \le H$ for all k, $x \in l(p)$ and $B > 0$ then

$$\sum_k |a_k x_k| \le B\left(\sum_k |a_k|^{q_k} B^{-q_k} + 1\right)\left(\sum_k |x_k|^{p_k}\right)^{1/H},$$

where q_k is an in Lemma 5 and a is any sequence.

§3 *Theorem* 1 — For a matrix $A = (a_{nk})$ and a full class Φ, the following are equivalent :

(a) $A \in (l_\infty \cap c_\Phi, c)$;

(b) $\lim_n \sum_{k \in S} a_{nk}$ exists for each $S \in \Phi$ and

$$\lim_n \sum_k a_{nk} \text{ exists ;} \qquad \ldots (3.1)$$

(c) (3.1) and the following hold;

$$\alpha_k = \lim_n a_{nk} \text{ exists for each } k \in N, \qquad \ldots (3.2)$$

$$H = \sup_n \sum_k |a_{nk}| < \infty \qquad \ldots (3.3)$$

and

$$\lim_n \sum_{k \in S} |a_{nk} - \alpha_k| = 0 \text{ for each } S \in \Phi. \qquad \ldots (3.4)$$

PROOF : Since the sequences $(\chi_s(k))$, $S \in \Phi$ and $e = (1, 1, 1, \ldots)$ are in c_Φ, the implication (a) $\Rightarrow$ (b) holds. Suppose that (b) holds. Since Φ is a covering hereditary class, it contains all singletons $\{k\}$, $k \in N$, so that (b) $\Rightarrow$ (3.2). Also (3.1) and (3.3) hold by hypothesis and Lemma 2(c) respectively. To prove (3.4), let $S \in \Phi$.

Define a matrix $A^s = (a^s_{nk})$ by

$$\left.\begin{array}{l} a^s_{nk} = a_{nk}, \quad k \in S \\ a^s_{nk} = 0 \text{ otherwise} \end{array}\right\} n \in N.$$

If x is any sequence of 0's and 1's with supp $x = S_1$, then

$$\lim_n \sum_k a^s_{nk} x_k = \lim_n \sum_{k \in S \cap S_1} a_{nk},$$

which exists by (b) since Φ is hereditary. By Lemma 1, $A^s \in (l_\infty, c)$. We note, by (3.2), that $\lim_n a^S_{nk}$ is α_k when $k \in S$ and 0 otherwise. By the wellknown characterization of (l_∞, c) (see Maddox[7] for instance), we then have

$$\lim_n \sum_{k \in S} | a^s_{nk} - \alpha_k | = \lim_n \sum_{k \in S} | a_{nk} - \alpha_k | = 0,$$

which is (3.4). Thus (b) $\Rightarrow$ (c).

Finally, let (c) hold. By (3.1)-(3.3), $A \in (c, c)$. If $x \in l_\infty \cap c_\Phi$ and $x = y + z$, $y \in c$, $z \in \delta_\Phi \cap l_\infty$, then $Ay \in c$ and S = supp $x \in \Phi$, so that

$$\left| \sum_k a_{nk} z_k - \sum_{k \in S} \alpha_k z_k \right| = \left| \sum_{k \in S} (a_{nk} - \alpha) z_k) \right|$$

$$\leq \sup_k |z_k| \sum_{k \in S} | a_{nk} - \alpha_k | \to 0, \text{ as } n \to \infty,$$

by (3.4). Hence $Az \in c$. Thus (c) $\Rightarrow$ (a) and the proof is complete.

Corollary 1 — A matrix $A = (a_{nk})$ sums every bounded statistically convergent sequence x to the sum stat-lim x iff A is regular and satisfies (3.4) with $\alpha_k = 0$ for each $k \in N$ and Φ consisting of all subsets of N having zero density.

Corollary 2 — If $p = (p_n)$, $q = (q_n)$ are nonnegative bounded sequences such that $r_n = p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0 > 0$ for all n and $\left(\frac{n}{r_n}\right) \in l_\infty$, then the (N, p, q) - matrix (Borwein[1]) sums every bounded statistically convergent sequence x to the sum stat-lim x.

This generalizes Theorem 5.11 of (Connor[2]).

Theorem 2 — If Φ is a full class of subsets of N, then $(\chi_\Phi, m(\phi)) = (l_\infty, m(\phi))$.

PROOF : In view of the obvious inclusion : $(l_\infty, m(\phi)) \subset (\chi_\Phi, m(\phi))$, it sufficies to show that $(\chi_\Phi, m(\phi)) \subset (l_\infty, m(\phi))$. Suppose that $A = (a_{nk}) \in (\chi_\Phi, m(\phi))$, that is,

$$\sup_{s \in N, \sigma \in C_s} \left| \frac{1}{\phi_s} \sum_{k \in S} \sum_{n \in \sigma} a_{nk} \right| < \infty \text{ for each } S \in \Phi. \quad \ldots (3.5)$$

If $A \notin (l_\infty, m(\phi))$, then by Lemma 3,

$$\sup_{s \in N, \sigma \in C_s} \frac{1}{q_s} \sum_{k} \left| \sum_{n \in \sigma} a_{nk} \right| = \infty.$$

Hence, for each $i \in N$, we can find $s_i \in N$ and $\sigma_i \in C_{s_i}$ such that

$$\frac{1}{\phi_{s_i}} \sum_{k} \left| \sum_{n \in \sigma_i} a_{nk} \right| > i. \quad \ldots (3.6)$$

Writing $b_{ik} = \frac{1}{\phi_{s_i}} \sum_{n \in \sigma_i} a_{nk}$, $i, k \in N$, (3.6) is the same as

$$\sum_{k} | b_{ik} | > i, \; i \in N,$$

which contradicts Lemma 2(c) since Φ is full and by (3.5), $\sup_i \left| \sum_{k \in S} b_{ik} \right| < \infty$ for each $S \in \Phi$. This completes the proof of Theorem 2.

Corollary 3 — A matrix transforms l_∞ into $m(\phi)$ iff it transforms all sequences of 0's and 1's with supports of zero density into $m(\phi)$.

Corollary 4 — A matrix $A = (a_{nk}) \in (\bar{c} \cap l_\infty, l_1)$

$$\text{iff} \sup_{s \in N, \sigma \in C_s} \sum_{k} \left| \sum_{n \in \sigma} a_{nk} \right| < \infty.$$

PROOF : This follows from Theorem 2 and Lemma 3, noting that $\chi_\Phi \subset \bar{c} \cap l_\infty \subset l_\infty$, Φ being the family of all subsets of N with zero density and $m(\phi) = l_1$ if $\phi_s = 1$ for all $s \in N$.

Remark 2 : By taking Φ to be the family of all subsets of N with uniform zero density[15], it can be shown as in Corollary 4 that $(\hat{c}, l_1) = (l_\infty, l_1)$.

The remaining theorems, namely, Theorems 3, 4, 5 include and unify several known and unknown results. Lemma 4 is crucial in these proofs.

Theorem 3 — If X is a closed subspace of l_∞, then $A = (a_{nk}) \in (c, X)$ iff (2.1) and the following hold :

$$(a_{nk})_n \in X \text{ for each fixed } k \in N \; ; \quad \ldots (3.7)$$

$$\left(\sum_{k} a_{nk} \right) \in X. \quad \ldots (3.8)$$

PROOF : The necessity of (3.7), (3.8) is routine work and that of (2.1) is a consequence of the wellknown characterization of (c, l_∞) and the inclusion : $(c, X) \subset (c, l_\infty)$. To prove sufficiency, let $x \in c$ and $\lim x = l$. Now

$$A_n x = \sum_k a_{nk} x_k = \sum_k a_{nk}(x_k - l) + l \sum_k a_{nk}. \qquad \text{... (3.9)}$$

By (2.1) and the fact that $x_k - l \to 0$, $\sum_k a_{nk}(x_k - l)$ converges uniformly in n. It follows then by Lemma 4, (3.7) and (3.8) that $Ax = (A_n x) \in X$ and the proof is complete.

Corollary 5 — A matrix $A = (a_{nk}) \in (c, \bar{c} \cap l_\infty)$ iff (2.1) holds alongwith (3.7), (3.8) with $X = \bar{c} \cap l_\infty$. Moreover, if these conditions hold and $x \in c$, $\lim x = l$, $a_{nk} \overset{\text{stat}}{\to} \alpha_k$, as $n \to \infty$, for each fixed k, $\sum_k a_{nk} \overset{\text{stat}}{\to} \alpha$, then stat-lim Ax = $l\alpha + \sum_k \alpha_k (x_k - l)$.

PROOF : Since $\bar{c} \cap l_\infty$ is closed in l_∞, the first part follows from Theorem 3. The second part is a consequence of hypotheses, (3.9) and Remark 1.

In the same way, we can prove :

Corollary 6 — A matrix $A = (a_{nk}) \in (c, \hat{c})$ iff (2.1) holds, (3.7) and (3.8) hold with $X = \hat{c}$. Under these conditions, if $x \in c$, $\lim x = l$ and $\left(\sum_k a_{nk}\right)$, $(a_{nk})_n$ are almost convergent to α and α_k, $k \in N$, respectively, then Ax is almost convergent to $l\alpha + \sum_k \alpha_k (x_k - l)$.

Theorem 4 — If X is a closed subspace of l_∞, then $A = (a_{nk}) \in (l(p), X)$ iff (3.7) holds along with the following :

(a) (2.3) holds when $1 < p_k \leq H < \infty$ for all k.

and

(b) (2.4) holds when $0 < p_k \leq 1$ for all k.

PROOF : The necessity of (2.3) and (2.4) follow from Lemma 5 in view of the inclusion : $(l(p), X) \subset (l(p), l_\infty)$ and that of (3.7) is routine work. We turn to the proof of sufficiency. By Lemma 4, it is enough to show that $\sum_k a_{nk} x_k$ converges uniformly under the hypothesis when $x \in l(p)$.

Consider case (a). If $x \in l(p)$, we can choose $m \in N$ such that $\sum_{k=m}^{\infty} |x_k|^{p_k} < 1$. By Lemma 6, we have

$$\sum_{k=m}^{\infty} |a_{nk} x_k| \le B \left(\sum_{k=m}^{\infty} |a_{nk}|^{q_k} B^{-q_k} + 1 \right) \left(\sum_{k=m}^{\infty} |x_k|^{p_k} \right)^{1/H}$$

$$\le B(M+1) \left(\sum_{k=m}^{\infty} |x_k|^{p_k} \right)^{1/H} \to 0 \text{ as } m \to \infty$$

uniformly in $n \in N$ which proves our assertion in this case.

If case (b) holds, then for $x \in l(p)$, we can choose $n_0 \in N$ such that $|x_k|^{p_k} < \frac{1}{k}$ for all $k \ge n_0$, where $K = \sup_{n,k} |a_{nk}|^{p_k}$, so that $|a_{nk} x_k| < 1$ for all $n \in N$ and all $k \ge n_0$, then for $m \ge n_0$,

$$\left| \sum_{k=m}^{\infty} a_{nk} x_k \right| \le \sum_{k=m}^{\infty} |a_{nk}|^{p_k} |x_k|^{p_k}$$

$$\le K \sum_{k=m}^{\infty} |x_k|^{p_k} \to 0, \text{ as } m \to \infty$$

uniformly in n.

The conclusion of Theorem 4 follows.

Corollary 7 — If the hypotheses of Theorem 4 hold with $X = \bar{c} \bigcap l_\infty$ and $a_{nk} \overset{\text{stat}}{\to} \alpha_k$ as $n \to \infty$, for each $k \in N$, then $\sum_k a_{nk} x_k \overset{\text{stat}}{\to} \sum_k \alpha_k x_k$ whenever $x \in l(p)$.

A similar conclusion holds when $X = \hat{c}$ (Nanda[9]).

Theorem 5 — Let X be a closed subspace of l_∞. The following conclusions hold:

(a) If $0 < p_k \le 1$ for all $k \in N$, then $A = (a_{nk}) \in (w(p), X)$ iff (3.7), (3.8) and the following hold :

there exists $B > 1$ such that

$$C \equiv \sup_n \sum_{r=0}^{\infty} \max_{2^r \le k < 2^{r+1}} [(2^r B^{-1})^{1/p_k} |a_{nk}|] < \infty. \qquad \text{... (3.10)}$$

(b) If $1 < p < \infty$, then $A = (a_{nk}) \in (W_p, X)$ iff (3.7), (3.8) and the following hold :

$$D = \sup_n \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_{k=2^r}^{2^{r+1}-1} |a_{nk}|^q \right)^{1/q} < \infty \left(\frac{1}{p} + \frac{1}{q} = 1 \right). \qquad \text{... (3.11)}$$

PROOF : The necessity of (3.7), (3.8) are obvious and those of (3.10), (3.11) can be sketched along the familiar lines of the proofs of Theorem 5 Lascarides and

Maddox[6] and Theorem 7(b) of Maddox[7] (p.172) respectively. We consider sufficiency. By Lemma 4, (3.8) and (3.9), it is enough to establish the uniform convergence of $\sum_k a_{nk}(x_k - l)$ when $x_k \to l(w(p))$ in case (a) and $x_k \to l(w_p)$, $1 < p < \infty$, in case (b).

First, let the conditions of (a) hold. If $x_k \to l(w(p))$, we can find $n_0 \in N$ such that for every $r \geq n_0$

$$2^{-r} \sum_{k=2^r}^{2^{r+1}-1} |x_k - l|^{p_k} < 1, \qquad \text{... (3.12)}$$

and then for $m \geq n_0$,

$$\sum_{k=2^m} |a_{nk}(x_k - l)| \leq \sum_{r=m}^{\infty} \sum_{k=2^r}^{2^{r+1}-1} |a_{nk}||x_k - l|$$

$$= \sum_{r=m}^{\infty} \sum_{k=2^r}^{2^{r+1}-1} [\,|a_{nk}|(2^r B^{-1})^{1/p_k}\,]\,[\,(2^{-r} B)^{1/p_k}|x_k - l|\,]$$

$$\leq B \sum_{r=m}^{\infty} \sum_{k=2^r}^{2^{r+1}-1} [\,|a_{nk}|(2^r B^{-1})^{1/p_k}\,]\,[\,2^{-r}|x_k - l|^{p_k}]$$

(by (3.12) and since $0 < p_k \leq 1$)

$$\leq B \sup_{r \geq m} \left[2^{-r} \sum_{2^r}^{2^{r+1}-1} |x_k - l|^{p_k} \right] \left[\sum_{r=m}^{\infty} \max_{2^r \leq k < 2^{r+1}} (2^r B^{-1})^{1/p_k} |a_{nk}| \right]$$

$$\leq BC \sup_{r \geq m} 2^{-r} \sum_{k=2^r}^{2^{r+1}-1} |x_k - l|^{p_k}, \text{ by (3.10)},$$

$\to 0$, as $m \to \infty$, which proves the assertion.

In case (b), the uniform convergence of $\sum_k a_{nk}(x_k - l)$ follows by Hölder's inequality and our hypotheses and this completes the proof of Theorem 5.

Remark : The case in which $p_k = p > 0$ and $X = \hat{c}$ in the above theorem is known (Nanda[9]). Lemma 4 will be found useful in extending many other known results in the same manner.

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