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SOME TRANSFORMATIONS FOR FUNCTIONS OF MATRIX ARGUMENTS

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Lauricella functions of several real symmetric positive definite matrix arguments are examined under various types of transformations involving the argument matrices. A number of results are obtained on Lauricella functions f_A and f_D showing the effect of certain transformations of the matrix variables.

1. INTRODUCTION

Hypergeometric functions of many matrix arguments appear in different disciplines and in many practical situations, some details of which may be seen from Mathai[1, 3, 4]. When evaluating integrals involving such functions, usually one has to resort to transformations of variables in order to simplify matters. When these functions occur in statistical distribution theory one often requires the distributions of functions of such argument matrices which result in transformations of variables (see Pederzoli[8]). Here we will examine the effect of some transformations on the hypergeometric functions of several matrix arguments of the Lauricella type. Lauricella functions of various types appear in a variety of statistical distribution problems. Some of these are pointed out in Mathai and Saxena[7].

All the matrices appearing in this article are $p \times p$ real symmetric positive definite unless stated otherwise. The following standard notations will be used. A prime denotes the transpose, that is X' is the transpose of the matrix X ; $X > 0$ means that the matrix X is positive definite. $0 < A < X < B$ means that $A = A' > 0$, $X = X' > 0$, $B = B' > 0$, $X - A > 0$, $B - X > 0$. $tr(.)$ denotes the trace of (.), while $|(.)|$ indicates the determinant of (.). $\int_A^B f(X)\, dX$ signifies that the scalar function $f(X)$ of the real symmetric positive definite matrix X is integrated over X such that $X = X' > 0$, $A = A' > 0$, $B = B' > 0$, $X - A > 0$, $B - X > 0$, and dX stands for the wedge product of the $p(p + 1)/2$ differential elements in X.

In general, when $X = X' > 0$ is a $p \times p$ real matrix, it is assumed that all the $p(p+1)/2$ real variables are functionally independent. When $A = A' > 0$, $A^{1/2}$ indicates the symmetric square root of A. A real matrix-variate gamma function will be defined as follows :

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4}\,\Gamma(\alpha)\,\Gamma\left(\alpha - \frac{1}{2}\right) \ldots \Gamma\left(\alpha - \frac{p-1}{2}\right) \qquad \ldots (1.1)$$

for $\text{Re}(\alpha) > \frac{p-1}{2}$, where Re (.) represents the real part of (.).

For deriving the results in this article we have used the result on the Jacobians of two basic transformations. These are available in the literature, see for instance, Mathai[2], and will be stated here as lemmas.

Lemma 1.1 — Let A and X be $p \times p$ nonsingular matrices where A is a constant matrix and $X = X'$ a symmetric matrix of $p(p+1)/2$ functionally independent real scalar variables. Then, ignoring the sign,

$$Y = AXA',\ X = X',\ |A| \neq 0 \Rightarrow dY = |A|^{p+1}\,dX. \qquad \ldots (1.2)$$

Lemma 1.2 — Let X be a nonsingular $p \times p$ symmetric matrix of $p(p+1)/2$ functionally independent real scalar variables. Then

$$Y = X^{-1} \Rightarrow dY = |X|^{-(p+1)}\,dX. \qquad \ldots (1.3)$$

Jacobians associated with the different transformations appearing in the results later on are evaluated by using Lemmas 1.1 and 1.2.

2. Some Transformations on f_A

Lauricella function f_A of the matrix arguments $X_j = X_j' > 0$, $j = 1, 2, \ldots, n$ can be defined in many ways, as pointed out for examples, in Mathai[1, 3, 4]. An integral representation will be used to define f_A here.

$$f_A = f_A(a, b_1, \ldots, b_n; c_1, \ldots, c_n; -X_1, \ldots, -X_n)$$

$$= \left\{\prod_{j=1}^{n} \frac{\Gamma_p(c_j)}{\Gamma_p(b_j)\,\Gamma_p(c_j - b_j)}\right\} \int_0^I \cdots \int_0^I |U_1|^{b_1 - (p+1)/2} \ldots |U_n|^{b_n - (p+1)/2}$$

$$\times |I - U_1|^{c_1 - b_1 - (p+1)/2} \ldots |I - U_n|^{c_n - b_n - (p+1)/2}$$

$$\times |I + X_1^{1/2} U_1 X_1^{1/2} + \ldots + X_n^{1/2} U_n X_n^{1/2}|^{-a}\, dU_1 \ldots dU_n \qquad \ldots (2.1)$$

for $\text{Re}(b_j) > \frac{p-1}{2}$, $\text{Re}(c_j - b_j) > \frac{p-1}{2}$, $j = 1, 2, \ldots, n$.

Transformations of the type $Y_j = (I + X_j)^{-1/2} X_j (I + X_j)^{-1/2}$, where $(I + X_j)^{-1/2}$ denotes the symmetric square root of the real symmetric positive definite matrix

$I+X_j$, will transform $X_j=X_j'>0$ to $Y_j=Y_j'>0$ and $0<Y_j<I$. If we transform

$$Z_j=(I+X_1+\ldots+X_n)^{-1/2}X_j(I+X_1+\ldots+X_n)^{-1/2}$$

then $X_j=X_j'>0$ will go to $Z_j=Z_j'>0$, $0<Z_j<I$ and further $0<Z_1+\ldots+Z_n<I$. These types of transformations are of interest when considering integration problems involving functions of several matrix arguments or when dealing with transformed variables in statistical distribution theory. We will examine the effect of such transformations on f_A and establish two sets of results here.

Theorem 2.1 — For f_A as defined in (2.1)

$$\begin{aligned} f_A &= f_A(a,b_1,\ldots,b_n;c_1,\ldots,c_n;-X_1,\ldots,-X_n)\\ &= |I+X_1|^{-a} f_A(a,c_1-b_1,b_2,\ldots,b_n;\ c_1,\ldots,c_n;\\ &\quad (I+X_1)^{-1/2}X_1(I+X_1)^{-1/2},\ -(I+X_1)^{-1/2}X_2(I+X_1)^{-1/2},\\ &\quad \ldots,\ -(I+X_1)^{-1/2}X_n(I+X_1)^{-1/2}). \end{aligned} \quad \ldots (2.2)$$

PROOF : For convenience let us denote the last factor in the integrand of (2.1) by g. That is,

$$\begin{aligned} g &= |I+X_1^{1/2}U_1X_1^{1/2}+\ldots+X_n^{1/2}U_nX_n^{1/2}|^{-a}\\ &= |I+X_1|^{-a}|I-(I+X_1)^{-1/2}X_1^{1/2}(X_1-U_1)X_1^{1/2}(I+X_1)^{-1/2}\\ &\quad +\ldots+(I+X_1)^{-1/2}X_1^{1/2}U_nX_n^{1/2}(I+X_1)^{-1/2}|^{-a}. \end{aligned} \quad \ldots (2.3)$$

Substitute (2.3) in (2.1) and make a change of the variable $I-U_1=V_1$ and interpret the integral as an f_A to see the result.

The result in (2.2) can also be looked upon as the effect of taking out the factor $|I+X_1|^{-a}$ from an f_A. By proceeding in the same way the following result can be established and therefore this will be stated without proof.

Theorem 2.2 — For f_A as defined in (2.1)

$$\begin{aligned} f_A &= f_A(a,b_1,\ldots,b_n;c_1,\ldots,c_n;-X_1,\ldots,-X_n)\\ &= |I+X_1+\ldots+X_n|^{-a} f_A(a,c_1-b_1,\ldots;c_n-b_n;c_1,\ldots,c_n;\\ &\quad (I+X_1+\ldots+X_n)^{-1/2}X_j(I+X_1+\ldots+X_n)^{-1/2},\ j=1,2,\ldots,n). \end{aligned} \quad \ldots (2.4)$$

Note that the result in Theorem 2.1 holds true for any X_j, $j=1,2,\ldots,n$. Hence there are $n=\binom{n}{1}$ results of this type.

Suppose now that U_i and U_j, for specific values of i and j are changed to $I - U_j = V_i$ and $I - U_j = V_j$ respectively in the expression (2.1). Then the resulting effect is that the factor $|I + X_i + X_j|^{-a}$ comes out and a result similar to the one in Theorem 2.1, with two of the parameters b_i and b_i changed to $c_i - b_i$ and $c_j - b_j$ respectively, would be obtained. The number of such possibilities is $\binom{n}{2} = n(n-1)/2$. If a given set of r of U_i's are changed to $(I - U_i)$'s, for example, $I - U_i = V_i, ..., I - U_r = V_r$, then the factor $|I + X_1 + ... + X_r|^{-a}$ comes out. There are of course $\binom{n}{r}$ such results. If all the U_j's are changed to $(I - U_i)$'s, then we get the result in Theorem 2.2. It follows that in this category there are $\binom{n}{1} + \binom{n}{2} + ... \binom{n}{n} = 2^n - 1$ results of which Theorems 2.1 and 2.2 represent only two instances. A similar set of $2^n - 1$ results are available by changing X_j to $-X_j, X_j > 0$ under the conditions, $\|X_j\| < 1$, $j = 1, 2, ..., n$ which gives $\binom{n}{1} = n$ results, $\|X_i + X_j\| < 1$, $i \neq j = 1, 2, ..., n$ which provides $\binom{n}{2}$ results and so on, until $\|X_1 + ... + X_n\| < 1$ which accounts for just one result, where $\|(.)\|$ denotes a norm of (.)

3. Some Transformations of f_D

For the various definitions of the Lauricella function f_D, one can consult Mathai[1,3,4]. Here f_D will be defined by the following integral.

$$f_D = f_D\ (a, b_1, ..., b_n; c; -X_1, ..., -X_n)$$

$$= \left\{ \frac{\Gamma_p(c)}{\Gamma_p(a)\,\Gamma_p(c-a)} \right\} \int_0^I |U|^{a-((p+1)/2)}\ |I-U|^{c-a-((p+1)/2)} \quad ... (3.1)$$

$$\times |I + UX_1|^{-b_1} ... |I + UX_n|^{-b_n}\, dU$$

for $X_j > 0$, $j = 1, 2, ..., n$, $0 < U < I$, $\mathrm{Re}(a) > \frac{p-1}{2}$, $\mathrm{Re}\ (c - a) > \frac{p-1}{2}$.

A few results about the effect of transformations of the vairables on f_D will be established here and stated as theorems. The first one is available by taking the integral over a general symmetric positive definite matrix rather than over U such that $0 < U < I$.

Theorem 3.1 — For f_D defined as in (3.1) and for $V = V' > 0$

$$f_D = f_D\,(a, b_1, \ldots, b_n; c; -X_1, \ldots, -X_n)$$

$$= \left\{ \frac{\Gamma_p(c)}{\Gamma_p(a)\,\Gamma_p(c-a)} \right\} \int_{V>0} |V|^{a-((p+1)/2)}\, |I+V|^{-c+b_1+\ldots+b_n}$$

$$\times\, |I + (I+X_1)\,V|^{-b_1} \ldots |I + (I+X_n)\,V|^{-b_n}\, dV. \qquad \ldots (3.2)$$

PROOF : Consider the transformation $U = (I+V)^{-1}\,V$ for $V = V' > 0$. Note that this is a symmetric transformation because $(I+V)^{-1}\,V = (I+V^{-1})^{-1} = V(I+V)^{-1}$, $(I+V^{-1})' = (I+V^{-1})$. Then

$$U = (I+V)^{-1}\,V \Rightarrow U^{-1} = I + V^{-1}$$

$$\Rightarrow |U|^{-(p+1)}\, dU = |V|^{-(p+1)}\, dV$$

$$\Rightarrow dU = |I+V|^{-(p+1)}\, dV.$$

Also note that

$$|I + (I+V)^{-1}\, VX_j|^{-b_j} = |I+V|^{b_j}\, |I + V(I+X_j)|^{-b_j};$$

$$|U|^{a-((p+1)/2)} = |V|^{a-((p+1)/2)}\, |I+V|^{-a+((p+1)/2)}$$

and

$$|I-U|^{c-a-((p+1)/2)} = |I+V|^{a-c+((p+1)/2)}.$$

It is sufficient to substitute these expressions in (3.1) to arrive at the desired result.

Theorem 3.2 — For f_D as defined in (3.1)

$$f_D = f_D\,(a, b_1, \ldots, b_n; c; -X_1, \ldots, -X_n)$$

$$= |I+X_1|^{-b_1} \ldots |I+X_n|^{-b_n} f_D\,(c-a; b_1, \ldots, b_n; c;$$

$$X_1^{1/2}\,(I+X_1)^{-1}\,X_1^{1/2}, \ldots, X_n^{1/2}\,(I+X_n)^{-1}\,X_n^{1/2}\,) \qquad \ldots (3.3)$$

PROOF : Change $U = I - V$ so that $dU = (-1)^{p(p+1)/2}\, dV$. Observe the following : for $V = V' > 0$, $X = X' > 0$, and $X^{1/2}$ denoting the symmetric square root of X, we can write for the determinant that

$$|I + (I-V)X| = |(I+X) - VX|$$

$$= |(I+X) - X^{1/2}\, VX^{1/2}|$$

$$= |I+X|\, \left| I - X^{1/2}\,(I+X)^{-1}\,X^{1/2}\, V \right|.$$

By taking out the terms $|I+X|^{-b_1}, ..., |I+X|^{-b_n}$ from (3.1) and reinterpreting the integral as an f_D, the result is obtained.

Note also that for any matrix $X = X' > 0$ one has $0 < X^{1/2}(I+X)^{-1}X^{1/2} < I$. The result in (3.3) may be looked upon as the effect of taking out the factor $|I+X_1|^{-b_1}, ..., |I+X_n|^{-b_n}$ from an f_D. It is possible to get a similar result for $f_D(a, b_1, ..., b_n; c : X_1, ..., X_n)$ as well; in this case the condition $\|X_j\| < 1$, $j = 1, 2, ..., n$ would be needed.

Theorem 3.3 — For the f_D as defined in (3.1)

$$f_D = f_D\,(a, b_1, ..., b_n; c\,; -X_1, ..., -X_n)$$

$$= |I+X_1|^{-a} f_D\,(a, c-b_1-\ldots-b_n, b_2, ..., b_n; c;$$

$$(I+X_1)^{-1/2}X_1(I+X_1)^{-1/2},\ I-(I+X_1)^{-1/2}(I+X_j)(I+X_1)^{-1/2},$$

$$j = 1, 2, ..., n). \quad ...\ (3.4)$$

PROOF : In the expression (3.2) put

$$W = (I+X_1)^{-1/2}V(I+X_1)^{1/2} \Rightarrow$$

$$dV = |I+X_1|^{-(p+1)/2}dW$$

and

$$V = (I+X_1)^{-1/2}W(I+X_1)^{-1/2}.$$

Note that

$$|V|^{a-((p+1)/2)} = |I+X_1|^{-a+((p+1)/2)}|W|^{a-((p+1)/2)}$$

$$|I+V|^{-c+b_1+\ldots+b_n} = |I+(I+X_1)^{-1}W|^{-c+b_1+\ldots+b_n}$$

$$|I+(I+X_1)V|^{-b} = |I+W|^{-b_1}$$

and for $j = 2, 3, ..., n$

$$|I+(I+X_j)V|^{-b_j} = |I+(I+X_1)^{-1/2}(I+X_j)(I+X_1)^{-1/2}W|^{-b_j}.$$

Now make the substitutions in the integrand of (3.2). The integral in (3.2), denoted by Q, reduces to the following expression :

$$Q = \int_{W>0} |W|^{a-(p+1)/2}|I+W|^{-b_1}|I+(I+X_1)^{-1}W|^{-c+b_1+\ldots+b_n}$$

$$\times\left\{\prod_{j=2}^{n}|I+(I+X_1)^{-1/2}(I+X_j)(I+X_1)^{-1/2}W|^{-b_j}\right\}dW. \quad ...\ (3.5)$$

Compare now this result with formula (3.2). The coefficient matrices are the following : for $j = 1$, let

$$I+Z_1 = (I+X_1)^{-1} \Rightarrow -Z_1 = I-(I+X_1)^{-1} = (I+X_1)^{-1/2}X_1(I+X_1)^{-1/2}$$

while for $j = 2, 3, ..., n$, let

$$I + Z_1 = (I + X_1)^{-1/2} (I + X_j) (I + X_1)^{-1/2}$$

$$\Rightarrow \ -Z_j = I - (I + X_1)^{-1/2} (I + X_j) (I + X_1)^{-1/2}$$

By interpreting the result as an f_D, the proof is complete.

Theorem 3.4 — For the f_D defined as in (3.1)

$$\begin{aligned} f_D &= f_D \ (a, b_1, ..., b_n; c; -X_1, ..., -X_n) \\ &= |I + X_1|^{c-a-b_1} |I + X_2|^{-b_2} \dots |I + X_n|^{-b_n} \\ &\quad \times f_D (c - a, c - b_1 - \dots - b_n, b_2, ..., b_n; c; \\ &\quad - X_1, I - (I + X_1)^{-1/2} (I + X_j) (I + X_1)^{1/2}, \ j = 2, 3, ..., n). \quad \dots (3.6) \end{aligned}$$

PROOF : In (3.2) take out $|(I + X_j) V|$ from all the factors, $j = 1, 2, ..., n$. From $|I + V|$ also take out $|V|$. Then the factor $\prod_{j=1}^{n} |I + X_j|^{-b_j}$ comes out and those containing $|V|$ and $|I + V|$ reduce to $|V|^{-(c-a)}$ and $|I + V|^{-c+b_1+\dots+b_n}$ respectively. Make the transformation

$$Y = (I + X_1)^{-1/2} V^{-1} (I + X_1)^{-1/2} \ \Rightarrow$$

$$dV = |I + X_j|^{-(p+1)/2} |Y|^{-(p+1)} dY$$

and

$$V^{-1} = (I + X_1)^{1/2} Y (I + X_1)^{1/2}.$$

The above operations bring about the following changes.

$$|I + (I + X_1)^{-1} V^{-1}|^{-b_1} \rightarrow |I + Y|^{-b_1}$$

$$|I + V^{-1}|^{-c+b_1+\dots+b_n} \rightarrow |I + (I + X_1) Y|^{-c+b_1+\dots+b_n}$$

and for $j = 2, 3, ..., n$

$$|I + (I + X_j)^{-1} V^{-1}|^{-b_j} \rightarrow |I + (I + X_1)^{1/2} (I + X_j)^{-1} (I + X_1)^{1/2} Y|^{-b_j}.$$

Writing

$$I + Z_j = (I + X_1)^{1/2} (I + X_j)^{-1} (I + X_1)^{1/2}, \quad j = 2, 3, ..., n$$

we have

$$-Z_j = I - (I + X_1)^{1/2} (I + X_j)^{-1} (I + X_1)^{1/2}$$

and

$$I + Z_1 = I + X_1 \ \Rightarrow \ -Z_1 = -X_1.$$

Substituting these in (3.2) and interpreting the outcome as an f_D, the result follows.

Note that instead of taking out X_1, the transformation could have been

$$Y = (I + X_k)^{-1/2} V^{-1} (I + X_k)^{-1/2}$$

for a specific value of k. Thus we could get n results of the type seen in Theorem 3.4. One can also look at Theorem 3.4 as representing the effect on f_D when taking out $\prod_{j=1}^{n} |I + X_j|^{-b_j}$ as well as an additional factor $|I + X_1|^{c-a}$. A similar result could also be obtained for $\|X_j\| < 1$, $j = 1, 2, \ldots, n$ by replacing X_j with $-X_j, X_j > 0$.

The results on f_A and f_D presented in this article can also be obtained for Lauricella functions of types f_B and f_C or for other hypergeometric functions of several matrix arguments. Other properties of this kind in the real scalar variable case may be seen from Srivastava and Karlsson[9]. Further types of results on Appell's and Humbert's functions have been obtained by Mathai[5], on Kampè de Feriet's functions by Mathai and Pederzoli[6].

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