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ASYMPTOTICS AND REPRESENTATION OF DEFICIENT CUBIC SPLINES*

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In the present paper we have obtained asymptotically precise estimates for $s^{(j)} - f^{(j)}$, $j = 0, 1$ in terms of Bernoulli polynomials where s is the deficient periodic cubic spline interpolant of function f matching at two inner points of each mesh interval. Precise estimates of global asymptotic behaviour are also derived.

1. INTRODUCTION

The convergence properties of periodic deficient cubic splines which interpolate to a given function at two inner points of the given mesh interval have been studied by Meir and Sharma[5]. Considering a function $f \in C^4$, Rosenblatt[6] has obtained asymptotically precise estimates of the difference between the derivative of the cubic spline interpolating at the mesh points and the derivative of the function interpolated which is sometimes used to smooth a histrograms (cf. Boneva *et al.*[3]). For further studies in this direction reference may be made to Dikshit and Rana[4] and Rosenblatt[7]. In the present paper we obtain similar precise estimates for periodic deficient cubic splines interpolating to a given function at two interior points of the given mesh when $f \in C^4$. Precise estimates of global asymptotic behaviour are also derived.

2. REPRESENTATION OF CUBIC SPLINE

Let $\Delta : 0 = x_0 < x_1 < \ldots < x_n = 1$ denote a partition of [0, 1] with equidistant mesh points x_i so that $h = x_i - x_{i-1} = 1/n$. Let π_m be the set of all real algebraic polynomials of degree at most m. We define the deficient polynomial spline class $S(m, \Delta)$ as,

$$S(m, \Delta) = \{s(x) \mid s(x) \in C^{m-2}[0, 1], \; s(x) \in \pi_m, \, x \in [x_{i-1}, x_i], \; i = 1, 2, \ldots, n\}.$$

Writing $t_i = x_{i-1} + h/3$, $y_i = x_{i-1} + 2h/3$ and considering a given function f we introduce the interpolatory conditions

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$$s(t_i) = f(t_i),\ s(y_i) = f(y_i),\ i = 1, 2, ..., n. \qquad ... (2.1)$$

Denoting $27(x-x_{i-1})^2\ \{3(x_i-x)+(x-x_{i-1})\}/13$ by R_i .a cubic spline interpolator $s(x)$ is given in terms of the slopes $m_i = s'(x_i)$, $i = 0, 1, ..., n$ by

$$\begin{aligned} 6h^2 s(x) = {} & [3h(x-x_{i-1})^2 - R_i + 8h^3/39]m_i \\ & + [-3h(x_i-x)^2 - R_i + 73h^3/39]\, m_{i-1} \\ & + 6R_i\, [f(y_i) - f(t_i)]/h + 6h^2\, [20 f(t_i) - 7f(y_i)]/13 \end{aligned} \qquad ... (2.2)$$

if $x \in [x_{i-1}, x_i]$. The continuity condition of $s(x)$ at the mesh points x_i, $i = 1, 2, ..., n-1$, leads to the equations

$$-(m_{i+1} - 11m_i + m_{i-1})/2 = F_i,\ i = 1, 2, ..., n-1, \qquad ... (2.3)$$

where

$$8hF_i = 180\, [f(t_{i+1}) - f(y_i)] - 63[f(y_{i+1}) - f(t_i)].$$

3. Estimation of the Inverse of the Coefficient Matrix

Through out the paper deficient cubic spline s under consideration is 1-periodic and without any loss of generality we assume that s satisfies the condition $s'(0) = 0$. Denoting the transposes of $[m_1, m_2, ..., m_{n-1}]$ and $[F_1, F_2, ..., F_{n-1}]$ by M and F, respectively, it may be observed that the system of equations (2.3) is written as

$$AM = F \qquad ... (3.1)$$

where the coefficient matrix A is a square matrix of order $n-1$. Following Ahlberg *et al.*[2], we estimate the inverse of A. For this we introduce the following square matrix of order n,

$$G_n(\alpha, \beta) = \begin{bmatrix} 2\beta & \alpha & 0 & \cdots & 0 & 0 & 0 \\ \alpha & 2\beta & \alpha & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha & 2\beta & \alpha \\ 0 & 0 & 0 & \cdots & 0 & \alpha & 2\beta \end{bmatrix}$$

where α, β are given real numbers such that $\beta^2 \geq \alpha^2$. It is easily seen that $|G_n|$ satisfies the following difference equation

$$|G_n(\alpha, \beta)| = 2\beta\, |G_{n-1}(\alpha, \beta)| - \alpha^2\, |G_{n-2}(\alpha, \beta)| \qquad ... (3.2)$$

with $|G_{-1}(\alpha, \beta)| = 0$, $|G_0(\alpha, \beta)| = 1$ and $|G_1(\alpha, \beta)| = 2\beta$ and for $\eta = (\beta^2 - \alpha^2)^{1/2}$,

$$2\eta\, |G_n(\alpha, \beta)| = (\beta + \eta)^{n+1} - (\beta - \eta)^{n+1},\ \ \beta^2 > \alpha^2$$

$$| G_n(\alpha, \beta) | = (n+1)\beta^n, \text{ otherwise.} \qquad \text{... (3.3)}$$

In order to determine the inverse of the coefficient matrix A we first observe that for $\alpha = -1/2$

$$b^{-n}(2\beta + r) | G_n(\alpha, \beta) | = 2\beta(1 - r^{2n}) + r(1 - r^{2n-2})/2 \qquad \text{... (3.4)}$$

where $\qquad r = (-1/2b) = -2[\beta - (\beta^2 - 1/4)^{1/2}]$.

Taking $2\beta = 11/2$ and $\alpha = -1/2$ in $| G_n(\alpha, \beta) |$, we see that the coefficient matrix A satisfies the following equation,

$$4 | A | = 22 | G_{n-2}(-1/2, 11/4) | - | G_{n-3}(-1/2, 11/4) |. \qquad \text{... (3.5)}$$

An application of (3.4) in (3.5) gives

$$2(11 + 2r)\, b^{2-n} | A | = (11 + r)^2 - r^{2n-6}(11r + 1)^2. \qquad \text{... (3.6)}$$

Now we are in position to write the elements $a_{i,j}$ of A^{-1} from the cofactors of the transpose matrix. Thus, for $0 < i \leq j \leq n - 2$ or $i = j = 0$ (cf. Ahlberg *et al.*[2], pp. 35-38),

$$| A | \, a_{i,j} = (b \cdot r)^{j-i} G_i(-1/2, 11/4)\, G_{n-j-2}(-1/2, 11/4) \qquad \text{... (3.7)}$$

and

$$| A | \, a_{0,j} = (b \cdot r)^j G_{n-j-2}(-1/2, 11/4) \text{ for } 0 < j \leq n - 2. \qquad \text{... (3.8)}$$

Thus, using (3.4)-(3.6), we have for $0 \leq i \leq j \leq n - 2$

$$(1 - r^2)(1 - r^{2n})\, a_{i,j} = -2r^{j+1-i}(1 - r^{2i+2})(1 - r^{2n-2j-2}).$$

From the above expression it may be seen easily that A^{-1} is symmetric. Let us consider a fixed value x, $0 < x < 1$. Then for fixed $\varepsilon > 0$ and $\varepsilon < i/n$, $j/n < 1 - \varepsilon$; the elements $a_{i,j}$ of A^{-1} can just be approximated asymptotically by $2r^{|j-i|}/(11 + 2r)$.

Thus, we have proved the following :

Theorem 3.1 — The coefficient matrix A of (3.1) is invertible and if $A^{-1} = (a_{i,j})$, then $a_{i,j}$ can just be approximated asymptotically as $n \to \infty$ by $2r^{|j-i|}/(11 + 2r)$ and

$$\sum_i \frac{2r^{|j-i|}}{(11 + 2r)} = \frac{2(1 + r)}{(1 - r)(11 + 2r)} \qquad \text{... (3.9)}$$

where $0 < \varepsilon < i/n$, $j/n < 1 - \varepsilon$ and $r = (3\sqrt{13} - 11)/2$.

Remark 3.1 : It is worthwhile to note that the estimate (3.9) is sharper than that obtained in terms of the infimum of the excess of the positive value of the leading diagonal element over the sum of the positive values of other elements in each row. For the later of these gives $\| A^{-1} \| \leq 2/9$ where as (3.9) shows that the $\| A^{-1} \|$ does not exceed (2/13).

Since A is invertible, it follows that there exists a unique periodic spline $s \in S(3, \Delta)$ which satisfies the interpolatory conditions (2.1).

4. Error Bounds

Considering throughout this section a 1-periodic function $f \in C^4$, we shall estimate the error function $e^{(j)} = s^{(j)} - f^{(j)}$, $j = 0, 1$, where s is the deficient cubic spline interpolant of f satisfying the interpolatory conditions (2.1). Considering the interval $[x_{i-1}, x_i]$ the derivative of the cubic spline interpolator is given by

$$h^2 s'(x) = [h(x - x_{i-1}) - T_i]\, m_i + [h(x_i - x) - T_i]\, m_{i-1} + 6T_i\, [f(y_i) - f(t_i)]/h \quad \text{... (4.1)}$$

where $T_i = 27(x_i - x)(x - x_{i-1})/13$.

Replacing m_i by $e'(x_i)$ in (4.1), we see that

$$h^2 s'(x) = [h(x - x_{i-1}) - T_i]\, e'(x_i) + [h(x_i - x) - T_i]\, e'(x_{i-1}) + R_i(f) \quad \text{... (4.2)}$$

where $\quad R_i(f) = [h(x - x_{i-1}) - T_i]\, f'(x_i) + [h(x_i - x) - T_i]\, f'(x_{i-1}) + 6T_i\, [f(y_i) - f(t_i)]/h.$

For $f \in C^4$, we have by Taylor's Theorem,

$$R_i(f) = h^2 f'(x) + f^{(4)}(x)\, h^2 (x_i - x)(x - x_{i-1})(2x - x_i - x_{i-1})/12 + O(h^5)$$

where x is an appropriate point in $[x_{i-1}, x_i]$ which is not necessarily the same at each occurrence. Rewriting eqn. (3.1) as

$$A(e'(x_i)) = (F_i) - A(f'(x_i)) = (H_i), \quad \text{... (4.3)}$$

say, we first estimate (H_i).

For since $f \in C^4$, we get the following by an application of Taylor's Theorem to the right-hand side of (4.3),

$$(H_i) = -h^3 f^{(4)}(x)/24 + O(h^3). \quad \text{... (4.4)}$$

Fix x, $0 < x < 1$, and let $x_i = [nx]/n$ where $[y]$ is the greatest integer less than or equal to y. Now noticing that $A^{-1} = (a_{i,j})$, we have from eqn. (4.3),

$$(e'(x_i)) = \Sigma_k\, a_{i,k} H_k$$

$$= \Sigma_{|k-i| \geq m} + \Sigma_{|k-i| < m} = (R_1) + (R_2)$$

say, where m is taken to be fixed but larger positive integer.

The expressions (R_1) and (R_2) are estimated separately. It is clear that $i \approx xn$ and $n - i \approx (1-x)n$ as $n \to \infty$. Now assuming that $f^{(4)}$ is monotonic and applying Abel's Lemma to the inner sums, we have for some positive constant K_1,

$$|(R_1)| \leq K_1 (0.15)^m h^2 \quad \text{... (4.5)}$$

by virtue of Theorem 3.1.

Next, we see that for the values of k occurring in (R_2),

$$|x_k - x| = O(h). \qquad \text{... (4.6)}$$

Thus, using the result of Theorem 3.1, (4.4) and the continuous differentiability of $f^{(4)}$, we have

$$\left| (R_2) - \sum_{|k-i|<m} \frac{2r^{|k-i|}}{(11+2r)} \left(-h^3 f^{(4)}(x)/24\right) \right| = O(h^3).$$

Since m is arbitrary, we get

$$(e'(x_i)) = -(1/156)\, h^3 f^{(4)}(x) + O(h^3). \qquad \text{... (4.7)}$$

Thus, we complete the proof of the following :

Theorem 4.1 — Let f be 1-periodic continuously differentiable up to fourth order. Let $s \in S(3, \Delta)$ be the cubic spline interpolant of f satisfying (2.1). Assume that x, $0 < x < 1$ is fixed and $i = [nx]$ is the greatest integer less than or equal to x with $x_i = i/n$. Then for $u = (x - x_{i-1})/h$,

$$s'(x) - f'(x) = -\frac{f^{(4)}(x)}{3!} h^3 \ [169B_3(u) + 27B_2(u) + 2]/169 + o(h^3) \ \text{... (4.8)}$$

and

$$s(x) - f(x) = -\frac{f^{(4)}(x)}{4!} h^4 \ [\{169B_4(u) + 36B_3(u) + 8B_1(u)\}/169$$

$$- 13/810] + O(h^4) \quad \text{... (4.9)}$$

where $B_k(u)$'s, ($k = 1, 2, 3, 4$) are Bernoulli polynomials with leading coefficient 1 (see Abramowitz and Segun[1]), so that

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{x}{2}$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

(4.7) along with (4.2) completes the proof of (4.8). Staring with the equation (2.2) and following closely the foregoing proof of (4.8), we get the relation (4.9).

Now following Rosenblatt[7] we shall obtain the estimates of the global measures of deviation

$$\int_0^1 [s^j(x) - f^j(x)]^2\, dx, \; j = 0, 1 \qquad \dots (4.10)$$

under the 1-periodicity of $s \in S(3, \Delta)$. The estimates (4.8)-(4.9) are uniformly valid away from the boundary points 0 and 1. In fact we shall prove the following :

Theorem 4.2 — If f be 1-periodic continuously differentiable up to fourth order. Let $s \in S(3, \Delta)$ be the 1-periodic deficient cubic spline interpolant of f satisfying the interpolatory conditions (2.1). Then for $h \to 0$

$$\int_0^1 [s'(x) - f'(x)]\, dx = \frac{0.21h^5}{7!} \int_0^1 [f^{(4)}(x)]^2\, dx + O(h^5) \qquad \dots (4.11)$$

and

$$\int_0^1 [s(x) - f(x)]^2\, dx = \frac{10^{-4}\, h^7}{6!} \int_0^1 [f^{(4)}(x)]^2\, dx + O(h^7). \qquad \dots (4.12)$$

The proof of Theorem 4.2 follows from Theorem 4.1 with a little computation when we appeal to the following relations (see Abramowitz and Segun[1])

$$\int_0^1 B_n(t)\, B_m(t)\, dt = (-1)^{n-1} \frac{m!\, n!}{(m+n)!} B_{m+n}(0), \quad m, n = 1, 2, \dots ,$$

and

$$\int_x^{x+1} B_n(t)\, dt = \frac{B_{n+1}(x+1) - B_{n+1}(x)}{n+1} = x^n, \; n = 0, 1, \dots .$$

In the case of nonuniform mesh one might expect a similar asymptotic result under the condition

$$\frac{\max(x_i - x_{i-1})}{\min(x_i - x_{i-1})} < d < \infty.$$

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