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DEGREE OF APPROXIMATION OF FUNCTIONS IN THE GENERALIZED HÖLDER METRIC

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We establish some sharper estimates of the approximation of function by the linear means of Fourier series in the generalized Hölder space.

1. DEFINITIONS AND NOTATIONS

Let $\omega(t)$ and $\omega^*(t)$ be two given moduli of continuity satisfying the following conditions (for $0 \le p < q \le 1$) :

$$\frac{(\omega(t))^{p/q}}{\omega^*(t)} = O(1) \quad (t \to +0). \qquad \dots (1.1)$$

Denote by $C_{2\pi}$ the space of all 2π-periodic continuous functions under sup norm and by H_ω the class of functions

$$H_\omega := \{f \in C_{2\pi} \mid |f(x) - f(y)| \le c\omega(|x-y|)\},$$

here and throughout the paper c is a positive constant but not the same at each appearance.

For $f \in H_\omega$, define the norm $\|\cdot\|_{\omega^*}$ by

$$\|f\|_{\omega^*} := \|f\|_C + \sup_{x,y} \left\{ \Delta^{\omega^*} f(x,y) \right\}, \qquad \dots (1.2)$$

where $\|f\|_C := \sup_{0 \le x \le 2\pi} |f(x)|$,

and $\Delta^{\omega^*} f(x,y) = \dfrac{|f(x) - f(y)|}{\omega^*(|x-y|)}, \quad (x \ne y)$

and $\Delta^0 f(x,y) = 0$. If $\omega(t) = c|t|^\alpha$ and $\omega^*(t) = c|t|^\beta$ $(0 \le \beta < \alpha \le 1)$, then the space

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$$H_\alpha = \{f \in C_{2\pi} \mid |f(x) - f(y)| \le c\,|x-y|^\alpha,\ 0 < \alpha \le 1\}$$

is a Banach space[4] and the metric induced by the norm

$$\|f\|_\beta = \|f\|_C + \sup_{x,y} \left\{ \frac{|f(x)-f(y)|}{|x-y|^\beta} \right\}$$

is called the Hölder metric.

Let $D = (d_{nk})$ $(k,\ n = 0,\ 1,\ \ldots)$ be a lower triangular infinite matrix of real numbers satisfying the following conditions :

(i) $d_{nk} \ge 0$ $(k,\ n = 0,\ 1,\ \ldots)$, $d_{nk} = 0$, $k > n$ and

$$\sum_{k=0}^{n} d_{nk} = 1 \qquad \ldots (1.3)$$

(ii) for every k, $d_{nk} \to 0$ $\quad (n \to \infty)$.

For $f \in C_{2\pi}$, its Fourier series is given by

$$\frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Define the linear means of Fourier series of $f(x)$ by

$$t_n(f, x) = \sum_{k=0}^{n} d_{nk}\, s_k(x), \quad n = 0,\ 1,\ \ldots \qquad \ldots (1.4)$$

where $s_k(x) = s_k(f, x)$ is the kth partial sum of Fourier series of f.

Throughout the paper we shall use the following notations :

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

$$D_{nk} = \sum_{r=n-k+1}^{n} d_{nr},$$

$$D_n(x) = \sum_{k=0}^{n} d_{nk} \sin\left(k + \frac{1}{2}\right) x / \sin\frac{1}{2} x,$$

$[x]$, the integral part of x.

2. Introduction

Prössdorf[4] and Mohapatra and Chandra[3] obtained some results on degree of approximation for the means (1.4) in the Hölder metric. Recently, Singh[5] established the following two theorems by means of some results of Chandra[1] and a mediate function $H(t)$ satisfying that

$$\int_t^{\pi} u^{-2}\,\omega(u)\,du = O(H(t)), \quad H(t) \geq 0 \qquad \text{... (2.1)}$$

and

$$\int_0^t H(u)\,du = O(tH(t)) \quad \text{as} \quad t \to +0. \qquad \text{... (2.2)}$$

Theorem A — Let $D = (d_{nk})$ satisfy conditions (i) and (ii) of (1.3) and $d_{nk} \leq d_{n,k+1}$ $(k = 0, 1, ..., n-1;\ n = 0, 1, ...)$. Then for $f \in H_\omega$, $0 \leq p < q \leq 1$

$$\|f(x) - t_n(f, x)\|_{\omega^*} = O\ [\{\omega(|x-y|)\}^{p/q}\ \{\omega^*(|x-y|)\}^{-1}]$$

$$\times\ \{(H(\pi/n))^{1-p/q}\, d_{nn}\,(n^{p/q} + d_{nn}^{-p/q})\}$$

$$+\ O(d_{nn}\, H(\pi/n)) \qquad \text{... (2.3)}$$

if $\omega(t)$ satisfies (2.1) and (2.2), and

$$\|f(x) - t_n(f, x)\|_{\omega^*} = O\ [\{\omega(|x-y|)\}^{p/q}\ \{\omega^*(|x-y|)\}^{-1}]$$

$$\times\ \{(\omega(\pi/n))^{1-p/q} + d_{nn}\, n^{p/q}\,(H(\pi/n))^{1-p/q}\}$$

$$+\ O\{\omega(\pi/n) + d_{nn}\, H(\pi/n)\}, \qquad \text{... (2.4)}$$

if $\omega(t)$ satisfies (2.1).

Theorem B — Let $D = (d_{nk})$ satisfy conditions (i) and (ii) of (1.3) and $d_{nk} \geq d_{n,k+1}$ $(k = 0, 1, ..., n-1;\ n = 0, 1, ...)$. And let $\omega(t)$ satisfy (2.1) and (2.2). Then for $f \in H_\omega$, $0 \leq p < q \leq 1$

$$\|f(x) - t_n(f, x)\|_{\omega^*} = O\ [\{\omega(|x-y|)\}^{p/q}\ \{\omega^*(|x-y|)\}^{-1}]$$

$$\times\ \{d_{n0}\,(H(d_{n0}))^{1-p/q}\,(n^{p/q} + d_{n0}^{-p/q})\}$$

$$+\ O(d_{n0}\, H(d_{n0})). \qquad \text{... (2.5)}$$

In this paper we shall establish some results on degree of approximation directly by means of modulus of smoothness $\omega_2(f, t)$, not by the mediate function $H(t)$. These estimates are sharper than the ones of Singh[5].

3. Main Results

Our main results are the following.

Theorem 3.1 — Let $D = (d_{nk})$ satisfy conditions (i) and (ii) of (1.3) and

$$d_{nk} \le d_{n,k+1} \quad (k = 0, 1, ..., n-1; \; n = 0, 1, ...).$$

And let $\omega(t)$, $\omega^*(t)$ satisfy (1.1). Then for $f \in H_\omega$,

$$\| f(x) - t_n(f, x) \|_{\omega^*} = O(1) \left(\sum_{k=1}^{n} D_{nk}/k \right)^{p/q} \left(\sum_{k=1}^{n} D_{nk}\, \omega_2(f, k^{-1})/k \right)^{1-p/q} \quad \text{... (3.1)}$$

Theorem 3.2 — Let $D = (d_{nk})$ satisfy conditions (i) and (ii) of (1.3) and

$$d_{nk} \ge d_{n,k+1} \quad (k = 0, 1, ..., n-1; \; n = 0, 1, ...).$$

And let $\omega(t)$, $\omega^*(t)$ satisfy (1.1). Then for $f \in H_\omega$,

$$\| f(x) - t_n(f, x) \|_{\omega^*} = O(1) \left(\sum_{k=0}^{n} d_{nk}\, \omega_2\left(f, \frac{1}{k+1}\right) \right)^{1-p/q} . \quad \text{... (3.2)}$$

4. Lemmas

We need the following Lemmas.

Lemma 4.1[6] — If d_{nk} is monotonic increasing for all $k = 0, 1, ..., n$. Then for $n^{-1} \le u \le \pi$,

$$| D_n(x) | \le (6\pi^2 + \pi)\, D_{n,(u^{-1})}/u, \quad \text{... (4.1)}$$

where $(u^{-1}) := \max \{1, [u^{-1}]\}$.

Lemma 4.2 — Let $\sigma_n(f, x) = (n+1)^{-1} \sum_{k=0}^{n} s_k(x)$ be the Fejér operator. Then we have

$$\| f(x) - \sigma_n(f, x) \|_C = O(1) \int_0^\pi \omega_2(f, t)\, F_n(t)\, dt = O(R_n), \quad \text{... (4.2)}$$

where $F_n(t)$ is the kernel of Fejér and $R_n := \frac{1}{n} \sum_{k=1}^{n} \omega_2(f, 1/k)$.

We omit the proof since it is similar to Theorem 7.10 of DeVore[2].

5. Proofs of Theorems

Proof of Theorem 3.1 — Set

$$E_n(x) = t_n(f, x) - f(x) = (2\pi)^{-1} \int_0^{\pi} \varphi_x(t) D_n(t)\, dt$$

and

$$E_n(x, y) = E_n(x) - E_n(y) = (2\pi)^{-1} \int_0^{\pi} (\varphi_x(t) - \varphi_y(t)) D_n(t)\, dt.$$

It is clear that

$$|\varphi_x(t) - \varphi_y(t)| \leq 4c\, \omega(|x - y|) \qquad \text{... (5.1)}$$

and

$$|\varphi_x(t) - \varphi_y(t)| \leq 2\omega_2(f, |t|). \qquad \text{... (5.2)}$$

Then, using (5.1), we get

$$|E_n(x, y)| \leq (2\pi)^{-1} \int_0^{\pi} |\varphi_x(t) - \varphi_y(t)|\, |D_n(t)|\, dt$$

$$= O(\omega(|x - y|)) \int_0^{\pi} |D_n(t)|\, dt \qquad \text{... (5.3)}$$

$$= O(\omega(|x - y|)) \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right)$$

$$:= O(\omega(|x - y|))(I_1 + I_2). \qquad \text{... (5.4)}$$

It is obvious that

$$I_1 \leq \int_0^{\pi/n} \frac{1}{\sin 1/2\, t} \sum_{k=0}^{n} d_{nk} |\sin(k + (1/2))t|\, dt$$

$$= O(1) \int_0^{\pi/n} \sum_{k=0}^{n} d_{nk}(k + (1/2))\, dt = O(1). \qquad \text{... (5.5)}$$

Using (4.1), we get

$$I_2 = O(1) \int_{\pi/n}^{\pi} D_{n, (t^{-1})}\, t^{-1}\, dt = O(1) \int_{1/\pi}^{n/\pi} t^{-1} D_{n, (t)}\, dt$$

$$= O(1) \sum_{k=1}^{n-1} \int_{k/\pi}^{(k+1)/\pi} t^{-1} D_{n, (t)}\, dt = O(1) \sum_{k=1}^{n-1} D_{n, k+1}/k$$

$$= O(1) \sum_{k=1}^{n} D_{nk}/k. \qquad \text{... (5.6)}$$

Observing that

$$\sum_{k=1}^{n} D_{nk}/k \geq \sum_{k=[n/2]}^{n} D_{nk}/k \geq D_{n,[n/2]} \sum_{k=[n/2]}^{n} \frac{1}{k} \geq \frac{1}{8}, \qquad \text{... (5.7)}$$

and combining (5.4)-(5.6), we obtain

$$|E_n(x,y)| = O(\omega(|x-y|)) \sum_{k=1}^{n} D_{nk}/k. \qquad \text{... (5.8)}$$

On the other hand, using (5.2), we have

$$|E_n(x,y)| = O(1) \int_0^{\pi} \omega_2(f,t)\,|D_n(t)|\,dt$$

$$= O(1) \left\{ \int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right\} \omega_2(f,t)\,|D_n(t)|\,dt := I_1' + I_2'. \qquad \text{... (5.9)}$$

Applying the similar method, we obtain

$$I_1' = O(1)\,\omega_2\left(f, \frac{1}{n}\right) \int_0^{\pi/n} |D_n(t)|\,dt = O(1)\,\omega_2\left(f, \frac{1}{n}\right), \qquad \text{... (5.10)}$$

and

$$I_2' = O(1) \int_{\pi/n}^{\pi} \omega_2(f,t)\,t^{-1} D_{n,(t^{-1})}\,dt$$

$$= O(1) \int_{1/\pi}^{n/\pi} \omega_2(f,t^{-1})\,t^{-1} D_{n,(t)}\,dt$$

$$= O(1) \sum_{k=1}^{n-1} \int_{k/\pi}^{(k+1)/\pi} \omega_2(f,t^{-1})\,t^{-1} D_{n,(t)}\,dt$$

$$= O(1) \sum_{k=1}^{n-1} \omega_2(f,k^{-1})\,D_{n,k+1}/k$$

$$= O(1) \sum_{k=1}^{n} D_{nk}\,\omega_2(f,1/k)/k. \qquad \text{... (5.11)}$$

Noting (5.7) and

$$\sum_{k=1}^{n} D_{nk}\,\omega_2(f, 1/k)/k \geq \omega_2(f, 1/n) \sum_{k=1}^{n} D_{nk}/k$$

$$\geq \frac{1}{8}\,\omega_2\left(f, \frac{1}{n}\right) \geq c\, I_1'$$

and combining (5.9)-(5.11), we obtain

$$|E_n(x, y)| = O(1) \sum_{k=1}^{n} D_{nk}\,\omega_2(f, 1/k)/k. \qquad \text{... (5.12)}$$

Therefore, using (5.8) and (5.12), it follows that

$$\max_{x, y} \frac{|E_n(x) - E_n(y)|}{\omega^*(|x-y|)}$$

$$= \max_{x, y} \left\{ \frac{|E_n(x) - E_n(y)|^{p/q}}{\omega^*(|x-y|)}\, |E_n(x) - E_n(y)|^{1-p/q} \right\}$$

$$= O(1)\left(\sum_{k=1}^{n} D_{nk}/k \right)^{p/q} \left(\sum_{k=1}^{n} D_{nk}\,\omega_2(f, 1/k)/k \right)^{1-p/q}. \qquad \text{... (5.13)}$$

Since

$$|E_n(x)| \leq \frac{1}{2\pi} \int_0^{\pi} |\varphi_x(t)|\,|D_n(t)|\,dt \leq \frac{1}{2\pi} \int_0^{\pi} \omega_2(f, t)\,|D_n(t)|\,dt,$$

from (5.9), we get

$$\| E_n(x) \|_C = O(1) \sum_{k=1}^{n} D_{nk}\,\omega_2(f, k^{-1})/k. \qquad \text{... (5.14)}$$

Finally, (3.1) follows from (5.13), (5.14) and (5.7). □

Proof of Theorem 3.2 — Applying the Abel transformation and noting $d_{nk} \geq d_{n, k+1}$ $(k = 0, 1, ..., n-1)$, we get

$$D_n(t) = \sum_{k=0}^{n} (k+1)\, F_k(t)\, (d_{nk} - d_{n, k+1})$$

$$= \sum_{k=0}^{n} (d_{nk} - d_{n,k+1}) \frac{\sin^2 \frac{1}{2}(k+1)t}{\sin^2 \frac{1}{2} t} \geq 0. \qquad \text{... (5.15)}$$

Hence, the kernel $D_n(t)$ of the means (1.4) is positive. Using (5.1) and (5.3), we obtain in view of (1.3)

$$|E_n(x,y)| = O(1)\,\omega(|x-y|) \int_0^{\pi} |D_n(t)|\,dt$$

$$= O(1)\,\omega(|x-y|) \int_0^{\pi} D_n(t)\,dt = O(\omega(|x-y|)). \qquad \text{... (5.16)}$$

On the other hand, from (5.2), (5.15) and (4.2), we have

$$|E_n(x,y)| = O(1) \int_0^{\pi} \omega_2(f,t)\,D_n(t)\,dt$$

$$= O(1) \sum_{k=0}^{n} (k+1)(d_{nk} - d_{n,k+1}) \int_0^{\pi} \omega_2(f,t)\,F_k(t)\,dt$$

$$= O(1) \sum_{k=0}^{n} (k+1)(d_{nk} - d_{n,k+1}) R_k$$

$$= O(1) \sum_{k=0}^{n} d_{nk}\,\omega_2\left(f, \frac{1}{k+1}\right). \qquad \text{... (5.17)}$$

Similarly, we have

$$|E_n(x)| = O(1) \int_0^{\pi} \omega_2(f,t)\,D_n(t)\,dt$$

$$= O(1) \sum_{k=0}^{n} d_{nk}\,\omega_2\left(f, \frac{1}{k+1}\right). \qquad \text{... (5.18)}$$

Finally, using the same method of proof of Theorem 3.1, (3.2) follows from (5.16)-(5.18). □

6. Remarks

(1) If $d_{nk} \leq d_{n,k+1}$ $(k = 0, 1, \ldots, n-1)$, then

$$\sum_{k=1}^{n} D_{nk}/k \le nd_{nn}$$

and

$$\sum_{k=1}^{n} D_{nk}\,\omega_2(f, k^{-1})/k \le d_{nn} \sum_{k=1}^{n} \omega(f, k^{-1}) \le d_{nn} \sum_{k=1}^{n} \omega(k^{-1}).$$

Since

$$\sum_{k=1}^{n} \omega(1/k) \sim \int_{\pi/n}^{\pi} t^{-2}\,\omega(t)\,dt = O(H(1/n)),$$

we have

$$\sum_{k=1}^{n} \left(\frac{D_{nk}}{k}\right)^{p/q} \left(\sum_{k=1}^{n} \frac{D_{nk}\,\omega_2(f, k^{-1})}{k}\right)^{1-p/q} = O(d_{nn})\, n^{p/q}\, (H(\pi/n))^{1-p/q}.$$

This shows that estimate (3.1) is better than (2.3) and (2.4).

(2) Now, we prove that under conditions of Theorem 3.2 we always have

$$\lim_{n\to\infty} \| f(x) - t_n(f, x) \|_{\omega^*} = 0.$$

In fact, let ε be an arbitrary given positive number, then there exists an integer N such that for every $k > N$,

$$\omega\left(\frac{1}{k+1}\right) \le \varepsilon.$$

Then,

$$\sum_{k=0}^{n} d_{nk}\,\omega\left(\frac{1}{k+1}\right) \le \sum_{k=0}^{N} d_{nk}\,\omega\left(\frac{1}{k+1}\right) + \varepsilon \sum_{k=N+1}^{n} d_{nk}$$

$$\le \omega(1) \sum_{k=0}^{N} d_{nk} + \varepsilon.$$

Since

$$\sum_{k=0}^{N} d_{nk} \to 0 \quad (n \to \infty),$$

we have

$$\| f(x) - t_n(f, x) \|_{\omega^*} = O(1) \left(\sum_{k=0}^{n} d_{nk}\, \omega\left(\frac{1}{k+1} \right) \right)^{1-p/q} \to 0 \ (n \to \infty).$$

(3) Now we give an example. Set

$$P_n = \sum_{k=0}^{n} \frac{1}{k+1} \sim \ln n,$$

and

$$d_{nk} = \frac{1}{(k+1)P_n} \quad (k = 0, 1, \ldots, n).$$

Using Theorem B, we have

$$\| \overline{N}_n(f, x) - f(x) \|_{\omega^*} = O(1) \left\{ \frac{1}{\ln n} \left(H\left(\frac{1}{\ln n} \right) \right)^{1-p/q} \times \left(n^{p/q} + \frac{1}{\ln^{p/q} n} \right) + \frac{1}{\ln n} H\left(\frac{1}{\ln n} \right) \right\}.$$

Observing that

$$\frac{n^{p/q}}{\ln n} \left(H\left(\frac{1}{\ln n} \right) \right)^{1-p/q} \geq c \frac{n^{p/q}}{\ln n} \left(\int_{1/\ln n}^{\pi} \frac{\omega(t)}{t^2} dt \right)^{1-p/q}$$

$$= c \frac{n^{p/q}}{\ln n} \left(\int_{1/\pi}^{\ln n} \omega\left(\frac{1}{t} \right) dt \right)^{1-p/q} \geq c n^{p/q} \omega\left(\frac{1}{\ln n} \right)$$

$$\geq c\, \omega(1) \frac{n^{p/q}}{\ln n} \to \infty \quad (n \to \infty,\ p > 0).$$

Hence, for this case we can not obtain useful estimate. On the other hand, if we use our Theorem 3.2, we have

$$\| \bar{N}_n(f,x) - f(x) \|_{\omega^*} = O(1) \left(\sum_{k=1}^{n} \frac{1}{(k+1)\ln n} \omega\left(f, \frac{1}{k}\right) \right)^{1-p/q}$$

$$= O(1) \left(\sum_{k=1}^{[\ln n]} + \sum_{k=[\ln n]+1}^{n} \right)^{1-p/q}$$

$$= O(1) \left(\frac{\ln \ln n}{\ln n} + \omega_2\left(f, \frac{1}{\ln n}\right) \right)^{1-p/q}$$

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