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ON COUNTABLY PARA-H-CLOSED SPACES

T. G. RAGHAVAN

Katraj House, Bank Colony, Vasanthanagar III st., Madurai 625 003

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A Hausdorff space is called countably para-H-closed if every countable open cover $\mathcal{U}$ of X admits a locally finite collection $\mathcal{V}$ of open subsets of X such that $\mathcal{V}$ refines $\mathcal{U}$ and $X = \bigcup \{cl\ V : V \in \mathcal{V}\}$. In this paper it is shown that the projective objects in the category of first countable Hausdorff countably para-H-closed spaces and continuous maps are discrete spaces.

The class of countably paracompact spaces was first introduced by Dowker[1]. Two generalizations of countably paracompact spaces were given by Singal and Arya[6, 7]. They were mildly paracompact and lightly paracompact spaces; the second one is also called countably para-H-closed spaces or almost countably paracompact spaces. Here in this paper we prefer the term 'countably para-H-closed spaces'. A topological space X is called countably paracompact if every countable open cover of X admits a locally finite refinement.

In his paper[5], the author studied locally countably para-H-closed spaces and their one-point countably para-H-closed extensions. It was proved that there exists a projective maximum in the set of all such one-point extensions. In the same paper[5], the author studied minimal countably para-H-closed spaces and minimal locally countably para-H-closed spaces; and proved that both such spaces are feebly compact. In this paper we characterize the projective objects in the category of first countable Hausdorff countably para-H-closed spaces and continuous maps. One may refer to Herrlich[3] for the definition of projective objects in a category.

MAIN THEOREM

Let $\mathcal{K}$ be the category of first countable Hausdorff countably para-H-closed spaces and continuous maps. The projective objects of $\mathcal{K}$ are precisely the discrete spaces.

All spaces treated in this paper are Hausdorff. 'nbd' stands for a neighbourhood. N stands for the set of all natural numbers. If τ is a topology on X and $A \subset X$, then τ/A stands for the relative topology of τ on A. τ-$cl\ A$ (τ-$int\ A$) represents the closure (interior) of A taken in (X, τ). We suppress τ and write simply $cl\ A$ ($int\ A$) when the context is clear.

Definition 1 (Singal and Arya[6]) — A space X is called countably para-H-closed (or lightly paracompact) if each countable open cover $\mathcal{U} = \{U_n : n \in N\}$ of X admits a X-locally-finite collection $\mathcal{V}$ of open subsets of X such that $\mathcal{V}$ refines $\mathcal{U}$ and $X = \bigcup \{cl\ V : V \in \mathcal{V}\}$.

Definition 2 (Raghavan[5]) — Let (X, τ) be a topological space and $A \subset X$. (i) A is called a countably para-H-closed set if $(A, \tau/A)$ is countably para-H-closed. (ii) A is called a cpH-set in X if every countable open cover $\mathcal{U}$ of A has a X-locally-finite collection $\mathcal{V}$ of open subsets of X such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup \{cl\ V : V \in \mathcal{V}\}$. (iii) X is called locally countably para-H-closed if every point of X has an open nbd whose closure is a cpH-set in X.

Countably para-H-closed sets, cpH-sets in X — these are defined in Hausdorff spaces which need not be first countable. However we will be considering these concepts in and for first countable Hausdorff spaces in several situations that follow. The context will make this clear. Also in this paper whenever we write Hausdorff(1)-space, it stands for a Hausdorff first countable space. Let us now collect a few following results (Propositions 3-7 and Theorem 8) that are necessary for our work here from the author[5].

Proposition 3 — Let X be a Hausdorff(1)-space and $A \subset X$. If A is a cpH-set in X, then A is closed. □

Proposition 4 — Let A be a regular closed subset of X. Then A is a countably para-H-closed set if and only if A is a cpH-set in X. □

Proposition 5 — The union of any finite collection of cpH-sets in X is a cpH-set in X. □

Proposition 6 — If A is a cpH-set in X, B is regular closed and $B \subset A$, then B is a cpH-set in X.

A description of one-point countably para-H-closed extensions of a locally countably para-H-closed Hausdorff(1)-space is given by the author[5]. We take the base space involved to be Hausdorff(1). However, while the extensions themselves are all Hausdorff, they may or may not be first countable. One may refer to Porter[4] for the definition of an extension, an extension being projectively larger (smaller) than another, the concept of an extension being isomorphic to another and the concept of projective maximum (minimum) in a class $\mathcal{E}$ of extensions.

Proposition 7 — Let (X, τ) be a Hausdorff(1)-space. If (Y, σ) is an one-point Hausdorff countably para-H-closed extension, then X is open in Y and (X, τ) is locally countably para-H-closed. □

Theorem 8 — Let (X, τ) be a Hausdorff(1)-space which is locally countably para-H-closed, but not countably para-H-closed. Let $X^* = X \cup \{\pi\}$ where $\pi \notin X$. Then

(i) if $\mathcal{G} = \{V : V \in \tau$ and $X - V$ is a cpH-set in $X\}$, $\mathcal{F}$ is the open filter generated by $\mathcal{G}$ and $\tau^{\wedge} = \tau \bigcup \{\{\pi\} \bigcup U : U \in \mathcal{F}\}$, $(X^*, \tau^{\wedge})$ is a Hausdorff space;

(ii) $(X^*, \tau^{\wedge})$ is an one-point countably para-H-closed (not necessarily first countable) extension of (X, τ);

(iii) if $\tau^* = \tau \bigcup \{\{\pi\} \bigcup U : U \in \tau$ and $X\text{-}int\ cl\ U$ is a cpH-set in $X\}$, then (X^*, τ^*) is a Hausdorff space;

(iv) (X^*, τ^*) is an one-point countably para-H-closed (not necessarily first countable) extension of (X, τ);

(v) $\tau^{\wedge} \subset \tau^*$; and

(vi) (X^*, τ^*) is a projective maximum in the set of all one-point countably para-H-closed extensions of (X, τ). □

In general, if (Y, σ) is an one-point countably para-H-closed extension of (X, τ) and $Y = X^* = X \bigcup \{\pi\}$, then $\sigma \subset \tau^*$.

As defined in Dugundji[2] a map $p : X \to Y$ is called perfect if it is a continuous closed surjection and each fiber $p^{-1}(y)$ $(y \in Y)$ is compact. It is well-known that if Y is compact Hausdorff and Z is Hausdorff and $p : Y \times Z \to Z$ is the projection on the second factor, then p is closed so that p is perfect. Now let us prove that

Theorem 9 — Let $p : X \to Y$ be a perfect map which is open. Then if Y is countably para-H-closed, so is X.

PROOF : Let $\{U_n : n \in N\}$ be any countable open cover of X. Then $p(X - \bigcup_{i=1}^{m} U_i)$ is closed in Y for each $m \in N$. Further $\{Y - p(X - \bigcup_{i=1}^{m} U_i) : m \in N\}$ is an open cover of Y. Let $V_m = Y - p(X - \bigcup_{i=1}^{m} U_i)$. Then $p^{-1}(V_k) \subset \bigcup_{i=1}^{k} U_i$ and $V_k \subset V_{k+1}$. Let $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ be a locally finite collection of open subsets of Y which refines $\{V_n : n \in N\}$. We can find a precise locally finite collection $\{S_n : n \in N\}$ of open subsets of Y such that $S_n \subset V_n$ for each $n \in N$ and $\bigcup \{cl\ S_n : n \in N\} = Y$. Let $C = \{p^{-1}(S_n) : n \in N\}$. $p^{-1}(S_k) \subset p^{-1}(V_k) \subset \bigcup_{i=1}^{k} U_i$. Let $C_k = \{p^{-1}(S_k) \bigcap U_i : i = 1, 2, ..., k\}$ and $\mathcal{D} = \bigcup \{C_k : k \in N\}$. $\mathcal{D}$ is an open refinement (not necessarily a cover of X) of $\mathcal{U}$. Further given any $x_0 \in X$, there is a nbd T of $p(x_0)$ which meets only finitely many sets S_n so that the nbd $p^{-1}(T)$ of x_0 meets at most finitely many members of $\mathcal{D}$. Notice that $p^{-1}(cl\ S_n) \subset cl\ p^{-1}(S_n)$. Further $\bigcup \{cl\ C : C \in C_k\} = cl\ p^{-1}(S_k)$. Thus $\bigcup \{cl\ D : D \in \mathcal{D}\} = \bigcup \{cl\ p^{-1}(S_n) : n \in N\} \supset \bigcup \{p^{-1}(cl\ S_n) : n \in N\} = X$. □

Theorem 10 — Let Y be compact Hausdorff. Let X be countably para-H-closed, then $X \times Y$ is also countably para-H-closed.

PROOF : Consider the projection $p : X \times Y \to X$. Clearly p is an open map which is perfect. The result now follows from Theorem 9. □

Theorem 11 — Let X be locally countably para-H-closed and Y be compact Hausdorff. Then $X \times Y$ is locally countably para-H-closed.

PROOF : Let $x \in X$. Then there is an open nbd U_x of x such that $cl\ U_x$ is a cpH-set in X. Then by Theorem 10 and Proposition 4, $cl\ U_x \times Y$ is countably para-H-closed. Indeed $U_x \times Y$ is an open nbd of $(x, y) \in X \times Y$ such that $cl(U_x \times Y) = cl\ U_x \times Y$ is countably para-H-closed. Thus $X \times Y$ is locally countably para-H-closed. □

Theorem 12 — Let X and Y be Hausdorff and $p : X \to Y$ an open perfect map. Then, if Y is locally countably para-H-closed, so is X.

PROOF : Let $x \in X$ and $y = p(x)$. Let U be an open nbd of y such that $cl\ U$ is a cpH-set. Then if $q = p \mid p^{-1}(cl\ U)$,

$$q : p^{-1}(cl\ U) \to cl\ U$$

is an open perfect map so that $p^{-1}(cl\ U)$ is countably para-H-closed. In fact $p^{-1}(U)$ is an open nbd of x such that $cl\ p^{-1}(U) \subset p^{-1}(cl\ U)$ so that $cl\ p^{-1}(U)$ is a cpH-set. Thus X is locally countably para-H-closed. □

Proof of the Main Theorem

That X is projective in $\mathcal{K}$ when X is discrete is clear.

Conversely, if (X, τ) is projective in $\mathcal{K}$, let us prove that X is discrete.

Suppose X is not discrete. Then there is an element $a \in X$ such that $\{\mathbf{a}\}$ is not open in X. Let $Y = X - \{\mathbf{a}\}$ and $N^* = N \cup \{\omega\}$ be the one-point compactification of N, the discrete set of natural numbers. Let π be an abstract point not in $Y \times N^*$ and let $A = (Y \times N^*) \cup \{\pi\}$. Let us consider a topology ρ on A defined as follows:

(i) if $z \in Y \times N^*$, the set of all basic ρ-open nbds of z is precisely the set of all open nbds of z in the product topology;

(ii) the basic ρ-open nbds of π are of the form $(O \times N) \cup \{\pi\}$ where $O \cup \{\mathbf{a}\}$ is an open nbd of $\mathbf{a}$ in X.

The resulting space is Hausdorff.

Now let us show that (A, ρ) is countably para-H-closed. Clearly Y is locally countably para-H-closed by Proposition 7, so that by Theorem 11, $Y \times N^*$ is locally countably para-H-closed. Let $B = Y \times N^*$. Let σ be the product topology on B. Then $\rho\text{-}cl\ B = A$, B is open in A and A is an one-point extension of (B, σ). We may take $A = B^* = B \cup \{\pi\}$.

Let us now start with a subset $O(\subset Y)$ such that $O \cup \{\mathbf{a}\}$ is an open nbd of $\mathbf{a}$ in X. We can think of (X, τ) to be one-point countably para-H-closed extension of the locally countably para-H-closed space $(Y, \tau/Y)$. Let $\tau/Y = \sigma_1$. If $Y^* = Y \cup \{\mathbf{a}\} = X$ is endowed with the projective maximum topology σ_1^* (refer Theorem 8) then $O \cup \{\mathbf{a}\} \in \sigma_1^*$. Hence $Y - \sigma_1\text{-}int\ \sigma_1\text{-}cl\ O$ is countably para-H-closed. Thus $A - \rho\text{-}int\ \rho\text{-}cl\ ((O \times N) \cup \{\pi\}) = (Y - \sigma_1\text{-}int\ \sigma_1\text{-}cl\ O) \times N^*$ is a cpH-set in B.

Let $\mathcal{U} = \{U_n : n \in N\}$ be a countable ρ-open cover of $B^* = A$. There exists a $\mathbf{k} \in N$ such that $\pi \in U_k$ and a basic ρ-open nbd of π, $(O \times N) \cup \{\pi\} \subset U_k$; further $B - \sigma\text{-}int\ \sigma\text{-}cl(O \times N)$ is a cpH-set in B. Let $\mathcal{U}_0 = \{U_n \cap B : n \in N\}$. Then $\mathcal{U}_0$ is a countable σ-open cover of B and hence of $B - \sigma\text{-}int\ \sigma\text{-}cl\ (O \times N)$. Thus there is an open collection $\mathcal{V}_0$ which is locally finite, refines $\mathcal{U}_0$ and $\sigma\text{-}cl(\bigcup \mathcal{V}_0) \supset B - \sigma\text{-}int\ \sigma\text{-}cl(O \times N)$. If we write $\mathcal{V}_1 = \{V - \sigma\text{-}cl(O \times N) : V \in \mathcal{V}_0\}$ and $\mathcal{V} = \mathcal{V}_1 \cup \{O \times N\}$, $\mathcal{V}$ is a ρ-open collection which is locally finite in A and refines $\mathcal{U}$. Further $A = \rho\text{-}cl(\bigcup \mathcal{V})$. Thus (A, ρ) is countably para-H-closed.

Let us introduce yet another topology ρ' on A as follows :

(i) if $z \in Y \times N^*$, the set of all ρ'-open nbds of z is the set of all open nbds of z in the product topology;

(ii) the basic ρ'-open nbds of π are of the form $(O \times N^*) \cup \{\pi\}$ where $O \cup \{\mathbf{a}\}$ is an open nbd of $\mathbf{a}$ in X.

The resulting space is Hausdorff. It is to be noted that

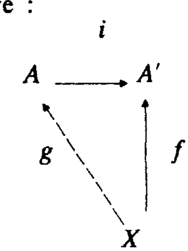
$$A - \rho\text{-}int\ \rho\text{-}cl\ ((O \times N^*) \cup \{\pi\}) = A - \rho'\text{-}int\ \rho'\text{-}cl\ ((O \times N^*) \cup \{\pi\})$$

and hence the previous arguments carry over to this case also so that (A, ρ') is countably para-H-closed.

Now $i : (A, \rho) \to (A, \rho')$ is continuous. Let us define $f : (X, \tau) \to (A, \rho')$ by taking

$$f(\mathbf{a}) = \pi$$
$$f(y) = (y, \omega) \text{ for each } y \in Y.$$

f is clearly continuous. Since X is projective, there is a continuous function g making the following diagram commutative :



where A stands for (A, ρ), A' for (A, ρ'). Thus $i \cdot g = f$ so that $g^{-1}((O \times N) \cup \{\pi\}) = \{\mathbf{a}\}$, a contradiction. □

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