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RECONSTRUCTION OF A GRAPH OF ORDER p FROM ITS $(p-1)$-COMPLEMENTS

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Let $G = (V, E)$ be a graph of order $p \geq 2$, and $P = \{V_1, V_2, ..., V_{p-1}\}$ be a partition of V of order $p-1$. The $(p-1)$-complement $G^P_{(p-1)}$ of G is obtained as follows: For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j, and add the missing edges between them. Disconnected graphs, trees, bipartite graphs and unicyclic graphs are reconstructible from the collection of all their $(p-1)$-complements.

1. INTRODUCTION

We consider only finite undirected graphs without loops and multiple edges, and follow the notation and terminology of the book by Behzad *et al.*[1].

Let $G = (V, E)$ be a graph of order $p \geq 2$, and k be an integer with $2 \leq k \leq p$. Suppose $P = \{V_1, V_2, ..., V_k\}$ is a partition of V of order k. The k-complement G^P_k of G is defined as follows : For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add the missing edges between them. The graph G is k-self complementary if for some partition P of V of order k, $G^P_k \cong G$. In Sampathkumar and Latha[4], k-self complementary trees, forests and unicyclic graphs have been characterized for all k, $2 \leq k \leq p$.

Clearly, $G^P_p \cong \overline{G}$, the complement of G, where P is the partition of V of order p. Now, consider a partition $P = \{V_1, V_2, ..., V_{p-1}\}$ of V of order $p-1$. Then, for exactly one V_i in P, $|V_i| = 2$ and for all V_j in P, $j \neq i$, $|V_j| = 1$. The graph $G^P_{(p-1)}$ is a $(p-1)$-complement of G.

A C^*-reconstruction of a graph G [from the collection of all its $(p-1)$-complements] is a graph H such that $V(G) = V(H)$, and both G and H have the same collection of $(p-1)$-complements. We say that G is C^*-reconstructible if every C^*-reconstruction of G is isomorphic to G.

Analogous to the well-known Ulam's conjecture (see Harary[3]) on reconstruction of graphs, one can state

Conjecture 1 — All graphs are C^*-reconstructible.

In this paper we show that disconnected graphs, unicyclic graphs, trees and bipartite graphs are C^*-reconstructible.

For a graph G, $G + x$ is a graph obtained from G by adding a new edge joining two nonadjacent vertices (if any) in G, and $G - x$ is the graph obtained from G by removing an edge of G.

Consider a $(p - 1)$-complement $G^P_{(p-1)}$ of G. Suppose $V_i = \{u, v\}$ is the set in P of cardinality two. If u and v are adjacent, then clearly $G^P_{(p-1)} \cong \overline{G} + uv$, and $\overline{G}^P_{(p-1)} \cong G - uv$.

Similarly, if u and v are nonadjacent in G, then $G^P_{(p-1)} \cong \overline{G} - uv$, and $\overline{G}^P_{(p-1)} \cong G + uv$.

Thus, the problem of reconstructing G from its $(p - 1)$-complements reduces to the problem of reconstructing G from the collection of all graphs $\{G + x, G - x\}$.

We observe that if a graph is C^*-reconstructible, then so is its complement.

Let C^+ and C^- respectively denote the collection of all graphs $G + x$ and $G - x$, and $C^* = C^+ \bigcup C^-$.

We shall adopt the following convention :

Suppose G is a graph of order $p \geq 2$. Then, $C^+ = \phi$ if and only if G is complete, and $C^- = \phi$ if and only if G is totally disconnected. Thus, $C^* = C^-$ if G is complete, and $C^* = C^+$ if G is totally disconnected.

2. Preliminary Results

We say that a graphical parameter or graphical property is C^*-recognizable if for each graph G, the value of the parameter for G or the knowledge of whether the graph G has the property in question can be determined from the collection C^*. The order and size of G are C^*-recognizable.

To start with, we derive some elementary results.

Proposition 1 — The classes C^+ and C^- in C^* can be recognized, and the size of G is C^*-recognizable.

Proof : *Case* 1 : Suppose all the graphs in C^* have the same number n of edges. Then, either $C^+ = \phi$ or $C^- = \phi$. Clearly, $n = 0$, if and only if $G = K_2$. In this case, $C^* = C^-$ and $C^+ = \phi$.

Also, $n = 1$ if and only if $G = rK_1$ for some integer $r \geq 2$. In this case, $C^* = C^+$ and $C^- = \phi$.

Further, $n = \binom{p}{2} - 1$ for some $p \geq 3$ if and only if $G = K_p$. In this case, $C^+ = \phi$.

Case 2 : Suppose Case 1 is not true. Then there exist exactly two classes of graphs in C^*, namely, the graphs having exactly $q + 1$ edges each and those having

$q - 1$ edges each, for some integer $q \geq 1$. These are respectively the classes C^+ and C^-. Clearly, q is the number of edges in G.

Corollary 1.1 — The complete graph K_p and the totally disconnected graph $\overline{K}_p$ are C^*-reconstructible.

One can easily reconstruct paths P_n and cycles C_n on n vertices.

Proposition 2 — (i) A graph G is P_n, $n \geq 2$ if and only if each graph in C^- is of the form $P_r \bigcup P_s$, where $r + s = n$.

(ii) A graph G is C_n, $n \geq 3$, if and only if each graph in C^- is P_n.

Also, regular graphs are C^*-reconstructible.

Proposition 3 — A graph G of order $p \geq 4$ is regular of degree $r < p - 1$ if and only if each graph in C^+ has exactly two adjacent vertices of degree $r + 1$, and the others have degree r.

For m, $n \geq 2$, let $K^2_{m,n}$ be the connected graph with exactly three blocks K_m, K_n and K_2 with K_m and K_n as end blocks, and K_2 as the middle block.

Proposition 4 — (i) $G = K_n \bigcup K_1$, $n \geq 2$, if and only if each graph in C^+ is K_n together with an end edge attached to one of its vertices.

(ii) $G = K_m \bigcup K_n$, $m, n \geq 2$, if and only if each graph in C^+ is $K^2_{m,n}$.

3. Reconstruction of Disconnected Graphs

We now reconstruct disconnected graphs. First, to determine connectedness of G, we have

Proposition 5 — Suppose $G \neq K_n \bigcup K_1$ or $K_m \bigcup K_n$, $m, n \geq 2$. Then, G is disconnected if and only if at least one of the graphs in C^+ is disconnected.

Further, if G is disconnected, the number of components in G is equal to the minimum number of components occurring in a graph in C^+ plus one.

Thus, connectedness and the number of components in G are C^*-recognizable.

To start with, we reconstruct graphs with isolated vertices.

Proposition 6 — A graph G of the form

(1) $rK_2 \bigcup sK_1$, $r \geq 1$, $s \geq 1$,

or (2) $K_n \bigcup sK_1$, $n \geq 3$, $s \geq 1$

is C^*-reconstructible.

Proof : (1) Clearly, every component of G is either K_2 or K_1 if and only if every component in any G_i in C^* is either P_4, P_3, K_2 or K_1.

Case 1 : $r = 1$ and $s \geq 1$. This is the case if and only if the collection C^- has exactly one graph $\overline{K}_p$ where $p = r + s + 1$.

Case 2 : $r \geq 2$ and $s \geq 1$. Clearly, $r \geq 2$ if and only if P_4 appears as a component in a G_i. In this case, the number s of isolated vertices in G is the number of such vertices in a G_i having P_4 as a component. After finding s, one can easily determine G from a G_i in C^*.

(2) Clearly, G is of the form $K_n \bigcup sK_1$, $n \geq 3$, $s \geq 1$, if and only if every G_i in C^- is of the form $K_n - x \bigcup sK_1$.

Proposition 7 — A disconnected graph with $c \geq 2$ components and $s \geq 1$ isolated vertices is C^*-reconstructible.

PROOF : If G is of the form $\overline{K}_n$, $rK_2 \bigcup sK_1$ or $K_n \bigcup sK_1$, $r \geq 1$, $s \geq 1$, $n \geq 2$, then G can be reconstructed as in Propositions 1, 4 and 5. Suppose G is not of any of the above types. The number c of components in G can be determined as in Proposition 5. Clearly, G has a component of order at least 3 if and only if there exists a G_i in C^* having c components, and having a component of order at least 3.

The number s of isolated vertices in G can be determined as follows :

If there exists a G_i in C^- having c components, then s is equal to the number of isolates in G_i. If not, there exists a G_i in C^+ having c components. Clearly, s is the number of isolates in G_i.

Case 1 : $s \geq 2$. Take a G_i in C^+ having $s - 2$ isolates. Clearly, K_2 appears as a component in G_i. In G_i, replace K_2 by $2K_1$ to get G.

Case 2 : $s = 1$. Take a G_i in C^+ having $c - 1$ components such that G_i contains a component having a 1-simple path of maximum length (a 1-simple path in a graph is a path whose one end vertex has degree 1, and the other end vertex has degree at least 1, and the other vertices on it, if any, have degree 2). Clearly, G is obtained from G_i by removing the edge incident to a pendant vertex of this path.

Proposition 8 — A disconnected graph G without isolates is C^*-reconstructible.

PROOF : The number c of components in G can be determined using Proposition 5.

If F is a component of maximum size s appearing in a graph G_i in C^-, then F is a component of G. We consider two cases.

Case 1 : F is complete. Let C_1^+ be the subcollection of C^+ consisting of all graphs with $c - 1$ components. Choose a H_i in C_1^+ with minimum number of components isomorphic to F. Then H_i has a component F_i having more edges than F. Clearly, F_i is got by connecting a component of G isomorphic to F, to another component of G by means of a bridge x. Since F is complete, with maximum size s, the bridge x can be easily recognized in H_i. Clearly, $H_i - x \cong G$.

Case 2 : F is not complete.

Among the graphs in C^+ having c components each, let G_j be one with minimum number of components isomorphic to F. Clearly, G_j has a unique component F_j of size $s + 1$. The graph G is obtained from G_j by replacing the component F_j with F.

Corollary 8.1 — Complement of a disconnected graph is C^*-reconstructible.

4. Reconstruction of a Tree

Trees can be recognized from the collection C^* as follows :

Proposition 9 — A graph G of order $p \geq 3$ is a tree if and only if each graph in C^- is a forest with exactly two components.

Proposition 10 — A tree G is C^*-reconstructible.

Proof : The number e of pendant vertices in G can be easily found. Clearly, e is the number of graphs in C^- having an isolated vertex. Now, take a graph H in C^+ with a triangle having two vertices of degree two. Indeed, such a graph exists in C^+. For, this is true if G has two pendant vertices adjacent to a common vertex. If not, clearly G has a vertex v of degree two adjacent to a pendant vertex, say u. If w is the other vertex adjacent to v, then uvw is a triangle in the graph $H = G + uw$, with $\deg u = \deg v = 2$. Since e is known, one can easily obtain G from H by removing an edge from the triangle in H.

Corollary 10.1 — Complement of a tree is C^*-reconstructible.

5. Reconstruction of a Bipartite Graph

It is easy to recognize a bipartite graph G from the collection C^*.

Proposition 11 — Suppose G is not a cycle. Then G is bipartite if and only if each $G - x$ is bipartite.

Proposition 12 — Bipartite graphs are C^*-reconstructible.

Proof : Let G be a bipartite graph. Since disconnected graphs and trees are reconstructible, we assume that G is neither disconnected nor a tree. Clearly, there exists a graph H in C^+ with a triangle got by adding an edge x to G joining two nonadjacent vertices on a cycle of G. Since the cycle in G is even, one can easily identify the edge x in H, the removal of which yields G.

Corollary 12.1 — Complement of a bipartite graph is C^*-reconstructible.

6. Reconstruction of a Unicyclic Graph

Unicyclic graphs can be recognized from the collection C^* as follows :

Proposition 13 — A graph G of order $p \geq 4$ is unicyclic if and only if G is connected, and each graph in C^+ has exactly two chordless cycles C_m and C_n, $m, n \geq 3$, with or without a common edge.

Proposition 14 — A unicyclic graph G of order $p \geq 4$ is C^*-reconstructible.

Proof : If G is a cycle, then it can be reconstructed as in Proposition 2. Otherwise, the length n of the cycle in G can be determined as follows :

Clearly, n is the length of the cycle of a graph in C^- having an isolated vertex. Also e, the number of pendant vertices in G is equal to the number of graphs in C^- having an isolated vertex.

Case 1 : $n \geq 4$. Take a graph $G_i = G + x$ in C^+ with a cycle of length n having a chord. Since we know n, we can easily identify the edge x in G_i, the removal of which gives G.

Case 2 : $n = 3$.

Subcase 2.1 : The order of G is four.

In this case G is K_3 together with a pendant edge at one of its vertices.

Subcase 2.2 : The order of G is five.

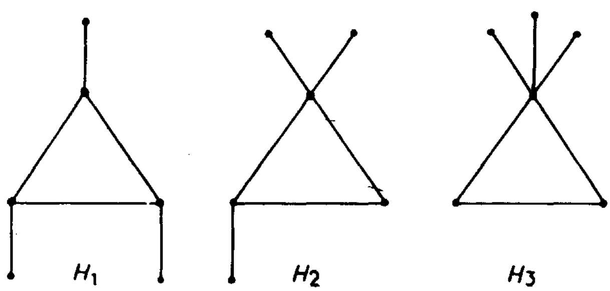
Clearly, in this case, $1 \leq e \leq 2$. If $e = 1$, then G is K_3 together with a path of length two attached to one of its vertices. If $e = 2$, there are two possibilities :

(i) G is K_3 together with two pendant edges at one of its vertices. This is so if and only if there exists a graph in C^+ which consists of two triangles with a common vertex.

The only other possibility is that G is K_3 with a pendant edge at exactly two of its vertices.

Subcase 2.3 : The order of G is six and $e = 3$.

Clearly G is one of the following types.



If there exists a graph in C^+ with two line disjoint triangles, then G is either H_2 or H_3, and G can be determined since $e = 3$. Otherwise $G = H_1$.

Subcase 2.4 : Suppose $G \neq H_i$, $1 \leq i \leq 3$, and the order of G is at least six.

One can find the number m of vertices of degree at least 3 on the cycle of G. It is equal to the maximum number of such vertices in a graph in C^- with an isolated vertex.

Subcase 2.4.1 : $2 \leq m \leq 3$.

Clearly there exists a graph in C^+ having exactly two triangles, exactly one of which has two vertices of degree two. Since the number e of pendant vertices in G is known, one can easily identify an edge x of this triangle, the removal of which yields G.

Subcase 2.4.2 : $m = 1$.

In this case, take a graph H in C^+ having two triangles with a common edge. There exists an edge x on one of the triangles in H such that the graph $H - x$ has a triangle with exactly one vertex having degree at least 3. Clearly, $H - x \cong G$.

Corollary 14.1 — Complement of a unicyclic graph is C^*-reconstructible.

Concluding remarks — The problem of reconstructing G from the collection C^- is precisely the edge reconstruction conjecture due to Harary[3].

A graph G with edge set $E(G) = \{e_1, e_2, ..., e_q\}$, $q \geq 1$, is said to be edge-reconstructible if for every graph H with $E(H) = \{f_1, f_2, ..., f_q\}$, $G - e_i \cong H - f_i$ for $i = 1, 2, ..., q$, implies that $G \cong H$. Thus, an edge reconstructible graph has the property that it is uniquely determined by the subgraphs $G - e$, $e \in E(G)$.

Not all graphs are edge-reconstructible. For example, the graphs $P_3 \bigcup K_1$ and $2K_2$, as well as the graphs $K_{1,3}$ and $K_3 \bigcup K_1$.

Note that no counter examples (if any) for our Conjecture 1 are known.

Edge Reconstruction Conjecture — Every graph of size at least four is edge-reconstructible.

We say that a graphical parameter or a graphical property is edge-recognizable if for each graph of size at least four, the value of the parameter for G or the knowledge of whether G has the property in question can be determined from the subgraphs $G - e$, $e \in E(G)$.

Since C^- is a subclass of C^*, it follows that if a parameter or a property of G is edge-recognizable, then it is C^*-recognizable.

It is known that the degree sequence of a graph of size at least four is edge-recognizable and hence is C^*-recognizable.

Also, if a graph G is edge-reconstructible, then it is C^*-reconstructible. It is known that several classes of graphs are edge-reconstructible : regular graphs, trees, disconnected graphs with at least two nontrivial components, etc. Thus, C^*-reconstructibility of these graphs can be deduced from edge-reconstructibility. But our results show that C^*-reconstructibility of these graphs is very much easier than edge-reconstructibility. Moreover, edge-reconstructibility of bipartite graphs is an open problem. As we have shown, bipartite graphs are easily C^*-reconstructible.

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