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EXISTENCE OF SOLUTIONS OF A DELAY DIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

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In this paper we prove the existence of a mild and a strong solutions of a delay differential equation with a nonlocal condition using the method of a compact semigroup and Schauder's fixed point principle.

1. INTRODUCTION

Byszewski[3] proved the existence and uniqueness of mild, strong and classical solutions of the following Cauchy problem :

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)), \quad t \in (t_0, t_0 + a],$$

$$u(t_0) + g(t_1, t_2, ..., t_p, u(\cdot)) = u_0, \qquad ... (*)$$

where $0 \leq t_0 < t_1 < ... < t_p \leq t_0 + a, \;\; a > 0,$

$-A$ is the infinitesimal generator of a C_0 semigroup on a Banach space X, $u_0 \in X$ and

$$f : [t_0, t_0 + a] \times X \to X, \;\; g : [t_0, t_0 + a]^p \times X \to X$$

are given functions. The symbol $g(t_1, t_2, ..., t_p, u(\cdot))$ is used in the sense that in the place of '$\cdot$' we can substitute only elements of the set $\{t_1, t_2, ..., t_p\}$.

The paper by Byszewski[3] was the first about nonlocal problems for the evolution equations in Banach spaces. His other works (see Byszewski[4, 5]) about the existence and uniqueness of solutions of abstract nonlocal Cauchy problem in Banach spaces are continuations of his earlier work[3]. Using semigroup method, Corduneanu[6] and Gripenberg *et al.*[7] proved the existence of solutions of Volterra integral equations of various types. The purpose of this paper is to study the existence of a mild and strong solution of the following nonlinear functional differential equation :

$$\frac{du(t)}{dt} + Au(t) = f(t, u(\sigma_1(t)), u(\sigma_2(t)), ..., u(\sigma_n(t))),$$

where $\sigma_i : [t_0, t_0 + a] \to [t_0, t_0 + a]$ $(i = 1, ..., n)$ are continuous functions with the nonlocal condition (*).

This paper is a generalization of the results of Balachandran and Ilamaran[1], Byszewski[3], Byszewski and Lakshmikantham[2] and Pazy[8].

2. Preliminaries

Consider the following nonlocal Cauchy problem :

$$\frac{du(t)}{dt} + Au(t) = f(t, u(\sigma_1(t)), u(\sigma_2(t)), ..., u(\sigma_n(t))), \quad \text{... (1)}$$

$$u(t_0) + g(t_1, t_2, ..., t_p, u(\cdot)) = u_0, \quad \text{... (2)}$$

where $0 \le t_0 < t_1 < ... < t_p \le t_0 + a$, $-A$ is the infinitesimal generator of a compact semigroup $T(t)$, $t \ge 0$, on a Banach space X and $f : I \times X^n \to X$, $g : I^p \times X \to X$, $\sigma_i : I \to I$ $(i = 1, ..., n)$ are the given continuous functions. Here $I = [t_0\ t_0 + a]$.

A continuous solution u of the integral equation

$$u(t) = T(t - t_0)\, u_0 - T(t - t_0)\, g(t_1, t_2, ..., t_p, u(\cdot)) + \int_{t_0}^{t} T(t - s) f(s, u(\sigma_1(s)), ..., u(\sigma_n(s)))\, ds,\ t \in I,$$

is said to be a mild solution of problem (1), (2) on I.

3. Existence of a Mild Solution

Theorem 1 — Assume that

(i) X is a Banach space with norm $\|\cdot\|$ and $u_0 \in X$;

(ii) $f : I \times X^n \to X$ is continuous in t on I and there exists a constant $L > 0$ such that

$$\| f(s, u_1, u_2, ..., u_n) \| \le L \text{ for } (s, u_1, u_2, ..., u_n) \in I \times X^n;$$

(iii) $g : I^p \times X \to X$ and there exists a constant $G > 0$ such that $G = \max_{u \in C(I,X)} \| g(t_1, t_2, ..., t_p, u(\cdot)) \|$;

(iv) $-A$ is the infinitesimal generator of a compact semigroup $T(t)$, $t \ge 0$, on X and $M := \max_{t \in [0, a]} \| T(t) \|$.

Then problem (1), (2) has a mild solution on I.

PROOF : Let $Y = C([0, a], X)$ and

$$Y_0 = \{u : u \in Y,\ u(t_0) + g(t_1, t_2, ..., t_p, u(\cdot)) = u_0, \| u \| \le r,\ 0 \le t \le a\},$$

where $r := M \| u_0 \| + MG + MLa.$

Clearly, Y_0 is a bounded closed convex subset of Y.

We define a mapping $F : Y \to Y_0$ by

$$(Fu)(t) = T(t - t_0)\, u_0 - T(t - t_0)\, g(t_1, t_2, ..., t_p, u(\cdot))$$

$$+ \int_{t_0}^{t} T(t - s) f(s, u(\sigma_1(s)), ..., u(\sigma_n(s)))\, ds,\ t \in I.$$

Since

$$\| (Fu)(t) \| \le \| T(t - t_0)\, u_0 \| + \| T(t - t_0)\, g(t_1, t_2, ..., t_p,\ u(\cdot)) \|$$

$$+ \int_{t_0}^{t} \| T(t - s) \| \,[\, \| [f(s, u(\sigma_1(s)), u(\sigma_2(s)), ..., u(\sigma_n(s))) \| ds$$

$$\le M \| u_0 \| + MG + MLa = r,$$

then F maps Y_0 into Y_0. Further , the continuiuty of F from Y_0 to Y_0 follows that f is continuous on $[0, a] \times X^n$. Moreover, F maps Y_0 into a precompact subset of Y_0. We prove that, the set $Y_0(t) := \{(Fu)(t) : u \in Y_0\}$ is precompact in X, for every fixed t, $0 \le t \le a$. Obviously for $t = t_0$, the set $Y_0(t_0) = \{u_0 - g\}$ is precompact. Let $t > t_0$ be fixed. Define

$$(F_\varepsilon u)(t) = T(t - t_0)\, u_0 - T(t - t_0)\, g(t_1, t_2, ..., t_p, u(\cdot))$$

$$+ \int_{t_0}^{t-\varepsilon} T(t - s) f(s, u(\sigma_1(s)), ..., u(\sigma_n(s)))\, ds$$

$$= T(t - t_0)\, u_0 - T(t - t_0)\, g(t_1, t_2, ..., t_p, u(\cdot))$$

$$+ T(\varepsilon) \int_{t_0}^{t-\varepsilon} T(t - \varepsilon - s) f(s, u(\sigma_1(s)), ..., u(\sigma_n(s)))\, ds \text{ for } t_0 < \varepsilon < t.$$

Since $T(t)$ is compact for every $t > t_0$, the set

$$Y_\varepsilon(t) := \{(F_\varepsilon u)(t) : u \in Y_0\}$$

is precompact in X for every ε, $t_0 < \varepsilon < t$.

Furthermore, for $u \in Y_0$ we have

$$\| (Fu)(t) - (F_\varepsilon u)(t) \| \leq \int_{t-\varepsilon}^{t} \| T(t-s) f(s, u(\sigma_1(s)), \ldots, u(\sigma_n(s))) \| ds \leq ML\varepsilon,$$

which implies that $Y_0(t)$ is totally bounded i.e. $Y_0(t)$ is precompact in X. We shall show that

$$F(Y_0) = S := \{Fu \ : \ u \in Y_0\}$$

is an equicontinuous family of functions.

For $t_0 < t < s$, we have

$$\begin{aligned} \| (Fu)(t) - (Fu)(s) \| \leq & \| (T(t-t_0) - T(s-t_0)) u_0 \| \\ & + \| (T(t-t_0) - T(s-t_0)) g(t_1, t_2, \ldots, t_p, u(\cdot)) \| \\ & + \| \int_{t_0}^{t} (T(t-\tau) - T(s-\tau)) f(\tau, u(\sigma_1(\tau)), \ldots u(\sigma_n(\tau))) d\tau \| \\ & + \| \int_{t}^{s} T(s-\tau) f(\tau, u(\sigma_1(\tau)) \ldots, u(\sigma_n(\tau))) d\tau \| \\ \leq & \| T(t-t_0) - T(s-t_0) \| \| u_0 \| + G \| T(t-t_0) - T(s-t_0) \| \\ & + L \int_{t_0}^{t} \| T(t-\tau) - T(s-\tau) \| d\tau + LM (s-t). \end{aligned}$$

The right-hand side of the above inequality is independent on $u \in Y_0$ and tends to zero as $s \to t$ (as a consequence of the continuity of $T(t)$ in the uniform operator topology for $t > 0$ which follows from the compactness of $T(t)$, $t > 0$). It is also clear that S is bounded in Y. Thus by Arzela-Ascoli's theorem, S is precompact. Hence by the Schauder fixed point theorem, F has a fixed point in Y_0 and any fixed point of F is a mild solution of (1), (2) on I such that $u(t) \in X$ for $t \in I$.

4. Existence of a Strong Solution

A function u is said to be a strong solution of problem (1), (2) on I if u is differentiable a.e on I, $u' \in L^1((t_0, t_0 + a]; X)$, $u(t_0) + g(t_1, t_2, \ldots, t_p, u(\cdot)) = u_0$

and

$$\frac{du(t)}{dt} + Au(t) = f(t, u(\sigma_1(t)), u(\sigma_2(t)), \ldots, u(\sigma_n(t))) \text{ a.e on } I.$$

Theorem 2 — Assume that :

(i) X is a reflexive Banach space with norm $\|\cdot\|$;

(ii) $f : I \times X^n \to X$ is continuous in t on I and there exist constants L and $L^* > 0$ such that $\|f(s, u_1, u_2, ..., u_n)\| \leq L$ and

$$\|f(s_1, u_1, u_2, ..., u_n) - f(s_2, v_1, v_2, ..., v_n)\|$$
$$\leq L^* \{\|s_1 - s_2\| + \|u_1 - v_1\| + \|u_2 - v_2\| + \dots + \|u_n - v_n\|\}$$

for $s, s_1, s_2 \in I;\ u_i, v_i \in X,\ (i = 1, \dots n)$;

(iii) $g : I^p \times X \to X$, there exists a constant $G > 0$ such that

$$G = \max_{u \in C(I,X)} \|g(t_1, t_2, ..., t_p, (\cdot))\|$$

and $g(t_1, t_2, ..., t_p, u(\cdot)) \in D(A)$;

(iv) $-A$ is the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$, on X and $M := \max_{t \in I} \|T(t)\|$;

(v) $u_0 \in D(A)$.

Then problem (1), (2) has a mild solution on I. Moreover, if there exists a unique mild solution u of problem (1), (2) on I such that

$$\|u(\sigma_i(s)) - u(\sigma_i(t))\| \leq R\,\|u(s) - u(t)\| \text{ for } s, t \in I,\ (i = 1, ..., n) \quad \dots (3)$$

where R is a positive constant, then u is a strong solution of problem (1), (2) on I.

PROOF : Since all the assumptions of Theorem 1 are satisfied then problem (1), (2) has a mild solution belonging to $C(I, X)$.

Assume now that u is a unique mild solution of problem (1), (2) on I such that inequalities (3) hold. We shall show that u is a strong solution of problem (1), (2) on I.

For any $t \in I$, we have

$$\begin{aligned} u(t+h) - u(t) = {} & [(T(t+h-t_0) - T(t-t_0))\, u_0] \\ & - [T(t+h-t_0)\, g(t_1, t_2, ..., t_p, u(\cdot)) \\ & - T(t-t_0)\, g(t_1, t_2, ..., t_p, u(\cdot))] \\ & + \int_{t_0}^{t_0+h} T(t+h-s) f(s, u(\sigma_1(s)), u(\sigma_2(s)), ..., u(\sigma_n(s)))\, ds \\ & + \int_{t_0+h}^{t+h} T(t+h-s) f(s, u(\sigma_1(s)), u(\sigma_2(s)), ..., u(\sigma_n(s)))\, ds \\ & - \int_{t_0}^{t} T(t-s) f(s, u(\sigma_1(s)), u(\sigma_2(s)), ..., u(\sigma_n(s)))\, ds \end{aligned}$$

$$= [(T(t+h-t_0) - T(t-t_0))u_0] - [T(t+h-t_0)\, g(t_1, t_2, ..., t_p, u(\cdot))$$
$$- T(t-t_0)\, g(t_1, t_2, ..., t_p, u(\cdot))]$$
$$+ \int_{t_0}^{t_0+h} T(t+h-s) f(s, u(\sigma_1(s)), u(\sigma_2(s)), ..., u(\sigma_n(s)))\, ds$$
$$+ \int_{t_0}^{t} T(t-s)\, [f(s+h, u(\sigma_1(s+h)), u(\sigma_2(s+h)), .. u(\sigma_n(s+h))$$
$$- f(s, u(\sigma_1(s)), u(\sigma_2(s)), ..., u(\sigma_n(s))]\, ds$$
$$= T(t-t_0)\, [T(h) - I]u_0 - T(t-t_0)\, [T(h) - I]g\, (t_1, t_2, ..., t_p, u(\cdot))$$
$$+ \int_{t_0}^{t_0+h} T(t+h-s) f(s, u(\sigma_1(s)), ..., u(\sigma_n(s)))\, ds$$
$$+ \int_{t_0}^{t} T(t-s)\, [f(s+h, u(\sigma_1(s+h)), u(\sigma_2(s+h)), .. u(\sigma_n(s+h)))$$
$$- f(s, u(\sigma_1(s)), u(\sigma_2(s)), ..., u(\sigma_n(s)))]\, ds.$$

According to our assumptions,

$$\| u(t+h) - u(t) \| \leq Mh \| Au_0 \| + Mh \| Ag(t_1, t_2, ..., t_p, u(\cdot)) \|$$
$$+ MLh + ML^* ha$$
$$+ ML^* \left\{ \int_{t_0}^{t} \| u(\sigma_1(s+h)) - u(\sigma_1(s)) \| \right.$$
$$+ \| u(\sigma_2(s+h)) - u(\sigma_2(s)) \|$$
$$\left. + ... + \| u(\sigma_n(s+h)) - u(\sigma_n(s)) \| \right\} ds$$
$$\leq Mh \| Au_0 \| + Mh \| Ag(t_1, t_2, ..., t_p, u(\cdot)) \| + MLh + ML^* ha$$
$$+ ML^* R \int_{t_0}^{t} n \| u(s+h) - u(s) \| ds$$
$$\leq Ph + ML^* Rn \int_{t_0}^{t} \| u(s+h) - u(s) \| ds.$$

where

$$P := M \| Au_0 \| + M \| Ag(t_1, t_2, ..., t_p, u(\cdot) \| + ML + ML^*a.$$

Using Gronwall's inequality, we get

$$\| u(t+h) - u(t) \| \leq Ph \exp (ML^*Rna), \quad t \in I.$$

Therefore, u is Lipschitz continuous on I.

The Lipschitz continuity of u on I combined with (ii) give that $t \to f(t, u(\sigma_1(t), u(\sigma_2(t)), ..., u(\sigma_n(t)))$ is Lipschitz continuous on I. Using Corollary 2.11 from Section 4.2 of Pazy[8] and using the definition of the strong solution, we obtain the linear Cauchy problem

$$\frac{dv(t)}{dt} + Av(t) = f(t, u(\sigma_1(t), u(\sigma_2(t)), ..., u(\sigma_n(t))), \ t \in [t_0, t_0 + a],$$

$$v(t_0) = u_0 - g(t_1, t_2, ..., t_p, u(\cdot)),$$

has a unique strong solution v satisfying the equation

$$v(t) = T(t-t_0)\, u_0 - T(t-t_0)\, g(t_1, t_2 ..., t_p, u(\cdot))$$

$$+ \int_{t_0}^{t} T(t-s) f(s, u(\sigma_1(s)), u(\sigma_2(s)), ..., u(\sigma_n(s)))\, ds = u(t), \quad t \in I.$$

Consequently, u is a strong solution of problem (1), (2) on I.

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