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ON THE EXISTENCE OF TWO-GENERATOR FINITE SOLVABLE GROUPS WITH SHORT DERIVED SERIES

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In this paper we prove the existence of some new infinite families of two-generator finite solvable groups with derived series of length ≤ 3.

1. INTRODUCTION

The Fuchsian groups (Macbeath[8]), in particular the Fuchsian triangle groups, play a very important role in the study of the automorphism groups of compact Riemann surfaces of genus ≥ 2. For example, the maximal automorphism groups of compact Riemann surfaces occur as quotients of a "special type" of a Fuchsian triangle group having periods 2, 3 and 7 (Macbeath[8]). Similarly the maximal solvable automorphism groups of compact Riemann surfaces also occur as quotients of another Fuchsian triangle group of periods 2, 3 and 8 (Chetiya[2]). On the other hand a quotient of a Fuchsian triangle group is a 2-generator group. This shows the importance of finding quotients of different classes of Fuchsian triangle groups from a strictly group theoretic point of view. This was the theme of the papers by Chetiya[3, 4], Chetiya and Kalita[5, 6] in which the existence of several infinite classes of two-generator finite solvable automorphism groups of compact Riemann surfaces was proved.

In this paper we consider another interesting class of Fuchsian triangle groups and construct infinitely many finite solvable quotients of these groups. Our results given in this paper supersede those of Chetiya, Chetiya and Kalita in the sense that those results come out as special cases of those of ours.

Our technique seems to be applicable in case we take a Fuchsian group of genus zero and 'any number' of periods, thus ensuring the possibility of proving the existence of infinitely many infinite classes of n-generator, $n \geq 3$, finite solvable groups.

2. PRELIMINARIES

An infinite group F generated by s elements $x_1, ..., x_s$ of finite orders $m_1, m_2, ..., m_s$ respectively, and $2g$ elements $a_1, b_1, ..., a_g, b_g$ of infinite order satisfying the relations

$$x_1^{m_1} = \ldots = x_s^{m_s} = \prod_{i=1}^{s} x_i \prod_{i=1}^{g} [a_i, b_i] = 1 \qquad \ldots (2.1)$$

where $[a_i, b_i]$ denotes the commutator of a_i, b_i is called a Fuchsian group if

$$\delta(F) = 2g - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right) > 0 \qquad \ldots (2.2)$$

and then F is said to have a presentation

$$\langle x_1, ..., x_s, a_1, b_1, ..., a_g, b_g : x_1^{m_1} = \ldots = x_s^{m_s} = \prod_{i=1}^{s} x_i \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$$

Such a Fuchsian group is usually denoted by $\Delta(g; m_1, ..., m_s)$. The non-negative integer g is called the 'genus' and the integers m_i (≥ 2) are called the 'periods' of F. If $g = 0$ then we denote F by $\Delta(m_1, ..., m_s)$. If $g = 0$ and $s = 3$, then $F = \Delta(m_1, m_2, m_3)$ is called a 'Fuchsian triangle group'. A Fuchsian group having no element of finite order except the identity is called a 'surface group'. A surface group K is generated by $2g$ elements of infinite order and is denoted by $\Delta(g;\ 0)$. Some properties of Fuchsian groups necessary for our purpose are stated below :

(I) If F_1 is a subgroup of F of finite index then F_1 is Fuchsian (Hoare *et al.*[7]) and

$$[F : F_1] = \frac{\delta(F_1)}{\delta(F)} \quad \text{(Macbeath[8])}.$$

(II) Let $F = \Delta(g; m_1, ..., m_s)$ be a Fuchsian group with generators $x_1, ..., x_s$ of finite order and $a_1, b_1, ..., a_g, b_g$ of infinite order. Let M be a 'normal' subgroup of F of 'finite index'. Denote by p_i the order of the image of x_i in the quotient group F/M and let $I = \{1 \leq i \leq s : m_i \neq p_i\}$. Now let $n_i = \frac{m_i}{p_i}$ and $s_i = \frac{[F:M]}{p_i}$ for every $i \in I$. Then $M = \Delta(g'; n_i, ..., n_i : i \in I)$ where n_i occurs s_i times and g' is obtained from (I). (Bujalance *et al.*[1], Maclachlan[9]).

3. EXISTENCE OF SOLVABLE QUOTIENTS

We now proceed to find an infinite family of solvable finite quotients of $\Delta(l, m, \mu l)$ where

(i) l, m, μ are integers ≥ 2

(ii) $(l, m) = (\mu, m) = 1$ and,

(iii) l, m, μ do not simultaneously assume the values $l = \mu = 2$, $m = 3$. [For any two positive integers k_1, k_2 the notation (k_1, k_2) is used to denote the h.c.f of k_1 and k_2].

We start with a definition followed by a lemma.

Definition 3.1 — Let K be any group. For each positive integer n, K_n^* is defined to be the subgroup of K generated by the nth powers of the elements of K, i.e.,

$$K_n^* = \langle \{x^n : x \in K\} \rangle.$$

Lemma 3.1 — Let K be a finitely generated group, K' its derived group, and K_n^* as defined above. Then $K' K_n^*$ is a characteristic subgroup of K and it is of finite index in K.

PROOF : Let K be generated by the elements $a_1, ..., a_t$. Then it is clear that K_n^* is a characteristic subgroup of K. Also, since the derived group K' of K is characteristic in K it now follows that $K'K_n^*$ is a characteristic subgroup of K. Further, since $K' \subseteq K'K_n^*$, $K/K'K_n^*$ is abelian. Now let $\alpha_1, ..., \alpha_t$ be the images of $a_1, ..., a_t$ under the natural homomorphism of K onto $K/K'K_n^*$. Then $K'K_n^*$ is generated by $\alpha_1, ..., \alpha_t$. Since $K_n^* \subseteq K'K_n^*$, the generators are connected by the relations

$$\alpha_1^n = ... = \alpha_t^n = 1.$$

It follows that $K/K'K_n^*$ is a finitely generated abelian group. And since all its generators are of finite order, it is isomorphic to a direct sum of a finite number of finite cyclic groups. Therefore $K/K'K_n^*$ is finite, i.e. $K'K_n^*$ is of finite index in K.

This completes the proof.

We now present the main result.

Theorem 3.1 — Let l, m, μ be integers ≥ 2 such that $(l, m) = (\mu, m) = 1$ and let l, m, μ do not simultaneously assume the values $l = \mu = 2$, $m = 3$. Then for each positive integer n, the Fuchsian triangle group $F = \Delta(l, m, \mu l)$ has a solvable quotient G_n of derived length ≤ 3, and of order $lm^{l-1} n^{2g} (n, \mu)^{m^{l-1}-1}$ where

$$g = 1 + \frac{1}{2} m^{l-2} (lm - 2m - l).$$

PROOF : Suppose F is generated by x_1, x_2, x_3 satisfying

$$x_1^l = x_2^m = x_3^{\mu l} = x_1 x_2 x_3 = 1$$

i.e. $$x_1^l = x_2^m = (x_1 x_2)^{\mu l} = 1.$$

Then F/F' is generated by u_1, u_2 where u_1 and u_2 are the images of x_1, x_2 respectively in F/F' under the abelianizing homomorphism and they satisfy the relations

$$u_1^l = u_2^m = (u_1u_2)^{\mu l} = 1, \; u_1u_2 = u_2u_1. \qquad \text{... (3.1)}$$

This implies $u_2 = 1$ whence

$$F/F' \cong Z_l.$$

From (3.1) and the fact that $(l, m) = (\mu, m) = 1$ it is easy to see that the orders of the images of x_1, x_2, x_3 in F/F' are l, 1 and l respectively. Also $[F : F'] = l$. Therefore, using (I) and (II) of §2 we see that

$$F' = \Delta(\underbrace{m, ..., m}_{l \text{ times}}, \mu)$$

As above if we consider F'/F'' we find that $[F' : F''] = m^{l-1}$ and

$$F'' = \Delta\,(g : \underbrace{\mu, ..., \mu}_{m^{l-1} \text{ times}})$$

where
$$2g - 2 + m^{l-1}\left(1 - \frac{1}{\mu}\right) = m^{l-1}\left[-2 + l\left(1 - \frac{1}{m}\right) + 1 - \frac{1}{\mu}\right]$$

i.e.
$$g = 1 + \frac{1}{2} m^{l-2}(lm - 2m - l).$$

Now, for each positive integer n let $K_n = F''' \; (F'')_n^*$. Then by Lemma 3.1 K_n is a characteristic subgroup of F'' and is of finite index in F''. Also, $F''' \subseteq K_n$. Since F'' is characteristic in F and K_n is characteristic in F'', it follows that K_n is characteristic and hence normal in F. Now, F'' is generated by the elments $y_1, ..., y_t, a_1, b_1, ..., a_g, b_g$ where $t = m^{l-1}$, satisfying the relations

$$y_1^\mu = \ldots = y_t^\mu = \prod_{i=1}^{t} y_i \prod_{j=1}^{g} [a_j, b_j] = 1.$$

Let $v_1, ..., v_t, \alpha_1, \beta_1, ..., \alpha_g, \beta_g$ be the images of $y_1, ..., y_t, a_1, b_1, ..., a_g, b_g$ respec- tively under the natural homomorphism of F'' onto F''/K_n. Then F''/K_n is generated by $v_1, ..., v_t, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, satisfying

$$v_1^\mu = \ldots = v_t^\mu = \prod_{i=1}^{t} v_i \prod_{j=1}^{g} [\alpha_j, \beta_j] = 1. \qquad \text{... (3.2)}$$

Since $F''' \subseteq K_n$, F''/K_n is abelian and therefore the generators of F''/K_n will commute with each other. Thereofore (3.2) reduce to

$$v_1^\mu = \ldots = v_t^\mu = v_1 \ldots v_t = 1. \qquad \text{... (3.3)}$$

Further, since $(F'')_n^* \subseteq K_n$ we have

$$v_1^n = \ldots = v_t^n = \alpha_1^n = \beta_1^n = \ldots = \alpha_g^n = \beta_g^n = 1. \qquad \text{... (3.4)}$$

From (3.3) and (3.4) it follows that the generators of F''/K_n satisfy the following relations :

$$\left.\begin{aligned} &\nu_1^{(n,\mu)} = \ldots = \nu_{t-1}^{(n,\mu)} = (\nu_1 \ldots \nu_{t-1})^{(n,\mu)} = 1 \\ &\alpha_1^n = \beta_1^n = \ldots = \alpha_g^n = \beta_g^n = 1. \end{aligned}\right\} \quad \ldots (3.5)$$

From (3.5) we conclude that

$$F''/K_n = \underbrace{Z_n \oplus \ldots \oplus Z_n}_{2g \text{ times}} \oplus \underbrace{Z_{(n,\mu)} \oplus \ldots \oplus Z_{(n,\mu)}}_{t-1 \text{ times}}.$$

Hence $$[F'' : K_n] = n^{2g}(n, \mu)^{t-1}$$
$$= n^{2g}(n, \mu)^{m^{l-1}-1}.$$

We now set $G_n = F/K_n$.

Then $G_n' = F'/K_n$, $G_n'' = F''/K_n$ and $G_n''' = \{1\}$ as $F''' \subseteq K_n$. Therefore G_n is a solvable group with the derived series

$$G_n \rhd G_n' \rhd G_n'' \rhd G_n''' = \{1\} \text{ and } G_n'' = G_n''' \text{ for } n = 1.$$

Further, $$|G_n| = |F/K_n|$$
$$= |F/F'|\,|F'/F''|\,|F''/K_n|$$
$$= [F : F']\,[F' : F'']\,[F'' : K_n]$$
$$= l\, m^{l-1}\, n^{2g}(n, \mu)^{m^{l-1}-1}.$$

This completes the proof.

Remark 1 : Every triple $(l, m, \mu l)$ satisfying the condition of Theorem 3.1 gives rise to an infinite family of finite solvable groups with derived series of length ≤ 3.

Remark 2 : In Theorem 3.1 we found that

$$F'' = (g; \overbrace{\mu, \ldots, \mu}^{m^{l-1} \text{ times}}) \text{ where}$$

$$g = 1 + \frac{1}{2} m^{l-2}(lm - 2m - l).$$

Now $g = 0$ only when

(i) $l = 2$, m is an odd integer relatively prime to μ and

(ii) $l = 3$, $m = 2$.

In the above cases F''' is of finite index in F'' and Lemma 3.1 may be applied on F''. We note that G_n obtained this way is solvable of derived length 4 (except for

$n = 1$ when it is solvable of derived length 3). In this way we can get the results of Chetiya[3, 4] and Chetiya and Kalita[5, 6].

Remark 3 : If we allow μ to take the value 1 in Theorem 3.1 excluding the cases when (i) $l = 2$ or (ii) $l = 3$, $m = 2$ our result remains valid. We note that here F'' is the surface group $\Delta(g; 0)$ where $g = 1 + \frac{1}{2} m^{l-2} (lm - 2m - l)$. Also, K_n, being a subgroup of a surface group, is a surface group. Then G_n is an automorphism group of a compact Riemann surface of genus $1 + n^{2g} (g - 1)$.

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