Indian J. pure appl. Math., **27**(6) : 575-579, June 1996

ON SOME DISCRETE WEYL-TYPE INEQUALITIES OF PACHPATTE

M. ALIĆ[1], C. E. M. PEARCE[2] AND J. PEČARIĆ[3]

[1]*Faculty of Mathematics, University of Zagreb, Zagreb*
[2]*Applied Mathematics Department, University of Adelaide, Adelaide, South Australia* 5005
[3]*Faculty of Textile Technology, University of Zagreb, Zagreb*

(*Received* 7 *August* 1995; *accepted* 29 *December* 1995)

Some discrete Weyl-type inequalities derived by Pachpatte are subsumed under a single, more general result.

1. INTRODUCTION

Weyl[6] has given the inequality

$$\int_0^\infty f^2\,dx \le 2\left[\int_0^\infty x^2 f^2\,dx\right]^{1/2}\left[\int_0^\infty f'^2\,dx\right]^{1/2}.$$

for a real-valued, continuously differentiable function on $(0, \infty)$ for which the integrals on the right exist. A more general version was presented by Hardy *et al.*[3].

This inequality proved important for applications and some effort has been put into extending it. Important generalizations were derived by Benson[1] and subsequently Bernis[3] and Pachpatte[4] have found multivariable variants.

Recently Pachpatte has discovered the existence of new discrete Weyl-type inequalities involving sequences $\{u_k\}$ of real numbers and their forward differences $\Delta u_k := u_{k+1} - u_k$.

The main result of Pachpatte[5] is given in the following theorem.

Theorem A — Let $\alpha \ge 0$, $p \ge 0$, $q \ge 1$ be real constants and $\{u_n\}_0^m$ a sequence of real numbers. Then

$$\sum_{n=0}^{m-1} n^{\alpha} |u_m|^{p+q} \le M \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha+q} |u_n|^{p} \left(\frac{|u_n|}{m} + |\Delta u_n| \right)^{q} \right\}^{1/q}$$

$$\times \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha} |u_n|^{p+q} \right\}^{1/q'}, \qquad \dots (1.1)$$

$$\sum_{n=0}^{m-1} n^{\alpha} |u_n|^{p+q} \le M \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha+1} |u_n|^{p} \left(\frac{|u_n|}{m} + |\Delta u_n| \right)^{q} \right\}^{1/q}$$

$$\times \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha+1} |u_n|^{p+q} \right\}^{1/q}, \qquad \dots (1.2)$$

$$\sum_{n=0}^{m-1} n^{\alpha} |u_n|^{p+q} \le M \left\{ \sum_{n=0}^{m-1} (n+1)^{(\alpha+1)q} |u_n|^{p} \left(\frac{|u_n|}{m} + |\Delta u_n| \right)^{q} \right\}^{1/q}$$

$$\times \left\{ \sum_{n=0}^{m-1} |u_n|^{p+q} \right\}^{1/q'}, \qquad \dots (1.3)$$

$$\sum_{n=0}^{m-1} n^{\alpha} |u_n|^{p+q} \le M \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha(q-1)} |u_n|^{p} \left(\frac{|u_n|}{m} + |\Delta u_n| \right)^{q} \right\}^{1/q}$$

$$\times \left\{ \sum_{n=0}^{m-1} (n+1)^{a(q'-1)+q'} |u_n|^{p+q} \right\}^{1/q'}, \qquad \dots (1.4)$$

where

$$M = \max\left\{ \frac{\alpha+2}{\alpha+1}, \frac{p+q}{\alpha+1} \right\} \text{ and } q' = \frac{q}{q-1}. \qquad \dots (1.5)$$

In this note we give a short derivation of a single inequality which subsumes (1.1)-(1.4) as special cases. In fact we derive slightly tighter versions of these four inequalities.

Minkowski's inequality is used to obtain a limiting form of our inequality for $m \to \infty$. This generalizes a corresponding result of Pachpatte.

2. Results

Theorem 1 — Let $\alpha \geq 0$, $p \geq 0$, $q \geq 1$ be real constants, $\{w_n\}_0^m$ a sequence of positive real numbers and $\{u_n\}_0^m$ a sequence of real numbers. Then

$$\sum_{n=0}^{m-1} n^\alpha |u_n|^{p+q} \leq M_m \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha+1} w_n^q |u_n|^p \left(\frac{|u_n|}{m} + |\Delta u_n| \right)^q \right\}^{1/q}$$

$$\times \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha+1} \frac{1}{w_n^{q'}} |u_n|^{p+q} \right\}^{1/q'} \quad \ldots (2.1)$$

where

$$M_m = \max\left\{ \frac{\alpha+2}{\alpha+1}, \frac{p+q}{\alpha+1} \cdot \frac{m-1}{m} \right\}.$$

Proof : As an intermediate result, Pachpatte proved the inequality

$$(\alpha+2) \sum_{n=0}^{m-1} (n+1)^{\alpha+1} \frac{v_n^r}{m} + \sum_{n=0}^{m-1} (n+1)^{\alpha+1}$$

$$\times \left(1 - \frac{n+1}{m}\right) r v_n^{r-1} (v_n - v_{n+1})$$

$$\geq (\alpha+1) \sum_{n=0}^{m-1} n^\alpha v_n^r$$

where $v_n = |u_n|$ and $r = p + q$.

From this we observe that

$$\sum_{n=0}^{m-1} n^\alpha |u_n|^r \leq \frac{\alpha+2}{\alpha+1} \sum_{n=0}^{m-1} (n+1)^{\alpha+1} \frac{|u_n|^r}{m}$$

$$+ \frac{r}{\alpha+1} \left(1 - \frac{1}{m}\right) \sum_{n=0}^{m-1} (n+1)^{\alpha+1} |u_n|^{r-1} |\Delta u_n|$$

$$\leq M_m \sum_{n=0}^{m-1} (n+1)^{\alpha+1} |u_n|^{r-1} \left(\frac{|u_n|}{m} + |\Delta u_n| \right)$$

$$= M_m \sum_{n=0}^{m-1} (n+1)^{\alpha+1} |u_n|^p \left[w_n \left(\frac{|u_n|}{m} + |\Delta u_n| \right) \right] \cdot \left[\frac{1}{w_n} |u_n|^{q-1} \right]$$

$$\leq M_m \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha+1} w_n^q |u_n|^p \left(\frac{|u_n|}{m} + |\Delta u_n| \right)^q \right\}^{1/q}$$

$$\times \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha+1} \frac{1}{w_n^{q'}} |u_n|^{p+q} \right\},$$

where in the last step we have used Hölder's inequality.

Remarks : With the choices $w_n \equiv (n+1)^{1/q'}, 1, (n+1)^{(\alpha+1)/q'}$ and $(n+1)^{\alpha-(2\alpha+1)/q}$ respectively we get (1.1), (1.2), (1.3) and (1.4) respectively, but with the smaller constant M_m in place of M. Thus we have in fact tighter inequalities than those of Theorem A.

Theorem 2 — Let $\alpha \geq 0, p \geq 0, q \geq 1$ be real constants, $\{w_n\}_0^\infty$ a sequence of positive real numbers and $\{u_n\}_0^\infty$ a sequence of real numbers. Then assuming the relevant sums exist

$$\sum_{n=0}^{\infty} n^\alpha |u_n|^{p+q} \leq M \left\{ \sum_{n=0}^{\infty} (n+1)^{\alpha+1} w_n^q |u_n|^p |\Delta u_n|^q \right\}^{1/q}$$

$$\times \left\{ \sum_{n=0}^{\infty} (n+1)^{\alpha+1} \frac{1}{w_n^{q'}} |u_n|^{p+q} \right\}^{1/q'}. \quad \ldots (2.2)$$

PROOF : By Minkowski's inequality, we have from (2.1) that

$$\sum_{n=0}^{m-1} n^\alpha |u_n|^{p+q} \leq M_m \left\{ \left[\sum_{n=0}^{m-1} (n+1)^{\alpha+1} w_n^q \frac{|u_n|^{p+q}}{m^q} \right]^{1/q} \right.$$

$$\left. + \left[\sum_{n=0}^{m-1} (n+1)^{\alpha+1} w_n^q |u_n|^p |\Delta u_n|^q \right]^{1/q} \right\} \left\{ \sum_{n=0}^{m-1} (n+1)^{\alpha+1} \frac{1}{w_n^{q'}} |u_n|^{p+q} \right\}^{1/q'}$$

$$\leq \frac{M}{m} \left\{ \sum_{n=0}^{\infty} (n+1)^{\alpha+1} w_n^q |u_n|^{p+q} \right\}^{1/q} \left\{ \sum_{n=0}^{\infty} (n+1)^{\alpha+1} \frac{1}{w_n^q} |u_n|^{p+q} \right\}^{1/q'}$$

$$+ M \left\{ \sum_{n=0}^{\infty} (n+1)^{\alpha+1} w_n^q |u_n|^p |\Delta u_n|^q \right\}^{1/q} \left\{ \sum_{n=0}^{\infty} (n+1)^{\alpha+1} \frac{1}{w_n^{q'}} |u_n|^{p+q} \right\}^{1/q'}$$

On letting $m \to \infty$ we get (2.2).

Remark : For $p = 0$ we have

$$\sum_{n=0}^{\infty} n^{\alpha} |u_n|^q \le M_1 \left\{ \sum_{n=0}^{\infty} (n+1)^{\alpha+1} w_n^q |\Delta u_n|^q \right\}^{1/q} \times \left\{ \sum_{n=0}^{\infty} (n+1)^{\alpha+1} \frac{1}{w_n^{q'}} |u_n|^q \right\}^{1/q'}$$

where

$$M_1 = \max \left\{ \frac{\alpha+2}{\alpha+1}, \frac{q}{\alpha+1} \right\}.$$

In the special case $\alpha = 0$, $q = 2$, $W_n \equiv 1/\sqrt{n+1}$ we recover the inequality

$$\sum_{n=0}^{\infty} |u_n|^2 \le 2 \left\{ \sum_{n=0}^{\infty} |\Delta u_n|^2 \right\}^{1/2} \left\{ \sum_{n=0}^{\infty} (n+1)^2 |u_n|^2 \right\}^{1/2}$$

which appears Weyl[5].

REFERENCES

1. D. C. Benson, *J. Math. Anal. Appl.* **17** (1967), 292-308.
2. S. Bernis, *Comm. Partial Differential equations* **9** (1984), 271-312.
3. G. H. Hardy, J. E. Litlewood and G. Pólya, *Inequalities*, C. U. P., Cambridge, 1934.
4. B. G. Pachpatte, *Tamkang J. Math.* **22** (1991), 259-65.
5. B. G. Pachpatte, *J. Math. Anal. Appl.* **188** (1994), 711-16.
6. H. Weyl, *The Theory of Groups and Quantum Mechanics* (English translation), Dover, New York, 1931.