Indian J. pure appl. Math., **27**(6) : 581-587, June 1996

IN REPLY TO THE QUESTION OF S. N. BERNSTEIN BY TRIGONOMETRIC INTERPOLATION POLYNOMIALS

JIAXING HE

Department of Mathematics, Jilin University of Technology, Changchun 130025, Jilin, People's Republic of China

(*Received* 2 *December* 1994; *after revision* 21 *November* 1995; *accepted* 2 *January* 1996)

In this paper, the third trigonometric interpolation summation polynomials $H_n(f; r, x)$ are constructed. If the function $f(x) \in C_{2\pi}$, then $H_n(f; r, x)$ converge the $f(x)$ on $(-\infty, \infty)$ uniformly; the convergence order is the best order if $f(x) \in C^j_{2\pi}$, $j \leq r$ (r is a nonnegative integer).

1. INTRODUCTION

In 1930, at the National Conference of Math. of U.S.S.R. in Kharkov, S. N. Bernstein put forward, for giving $\lambda\,(1 < \lambda < 2)$, and an arbitrary continuous function $f(x)$, if it is possible to construct a polynomial of interpolation with the degree of M ($< \lambda E$), it obtains the same value as $f(x)$ at given E points and it converges to $f(x)$ uniformly when $E \to \infty$. Naturally, what will be interesting is how to construct the trigonometric interpolation polynomial satisfying the S. N. Bernstein's requirement as above so that its convergence order is the best. In this paper, a new trigonometric polynomial of interpolation is constructed to answer the S. N. Bernstein problem.

Let $f(x) \in C_{2\pi}$ and

$$x_k^{(n)} = \frac{2k+1}{2n}\pi, \quad k = 0, 1, \ldots, 2n \qquad \ldots (1)$$

$\mu_k^{(n)}(x)$ is the fundamental trigonometric polynomial of Lagrange interpolation based on the nodes $x_k^{(n)}$, and

$$\mu_k^{(n)}(x) = \frac{1}{2n} \sin n(x - x_k^{(n)})\, ctg\, \frac{1}{2}(x - x_k^{(n)}), \quad k = 0, 1, \ldots, 2n. \qquad \ldots (2)$$

The nodes $x_0^{(n)}, x_1^{(n)}, \ldots, x_{2n}^{(n)}$ are divided into several groups with given even $2l$. The constructed trigonometric interpolation polynomial $H_n(f; r, x)$ (where r is a nonnegative integer) achieves the same value as $f(x)$ at the $2l-1$ nodes of each group and at $2lt$th node $H_n(f; r, x)$ is equal to A_{2lt-1}, where

$$A_{2lt-1} = f(x_{2lt-1}^{(n)}) + B_{2lt-1}, \quad t = 1, 2, \ldots, s. \qquad \ldots (3)$$

Let $h = \frac{\pi}{n}$ and

$$\sigma_h^{r+1} f(x) = \sum_{j=0}^{r+1} (-1)^j \binom{r+1}{j} f(x + (j-1)h) \qquad \ldots (4)$$

where $\binom{r+1}{j} = \frac{(r+1)!}{j!\,(r+1-j)!}$, then

$$B_{2lt-1} = \frac{1}{2^{r+1}} \sum_{i=1}^{2l} (-1)^i \sigma_h^{r+1} f(x_{2l(t-1)-1+i}^{(n)}) \qquad \ldots (5)$$

here r is a nonnegative integer. Thus

$$H_n(f; r, x) = \sum_{k=0}^{2n} A_k \mu_k^{(n)}(x) \qquad \ldots (6)$$

where when $k = 2lt - 1$, A_k is given by (3), the rest A_k is equal to $f(x_k^{(n)})$, $2n + 1 = 2ls + q$, $0 < q < 2l$.

For $H_n(f; r, x)$, we get :

Theorem 1 — Let $f(x) \in C_{2\pi}$. Then

$$\lim_{n \to \infty} H_n(f; r, x) = f(x)$$

is valid on $(-\infty, \infty)$ uniformly.

Theorem 2 — Let $f(x) \in C_{2\pi}^j$, $j \le r$. Then

$$|H_n(f; r, x) - f(x)| = O\left(E_n^*(f) + \frac{1}{n^j}\,\omega\left(f^j, \frac{1}{n}\right)\right) \qquad \ldots (7)$$

where O is independent of x, n, f, ..., f^j, $E_n^*(f)$ is the minimum deviation with the trigonometric polynomial of H_n^T to approximate the function $f(x)$, $\omega(f^j, \delta)$ is the modulus of continuity of $f^{(j)}(x)$ on $[0, 2\pi]$.

2. Lemma

Lemma — The following estimations are valid

$$\frac{1}{2^{r+1}} \sum_{k=0}^{2n} \sum_{m=0}^{r+1} \binom{r+1}{m} |\mu_k^{(n)}(x) + (-1)^m \mu_{k+1-m}^{(n)}(x)| = O(1) \qquad \ldots (8)$$

$$\sum_{t=1}^{s}\sum_{p=1}^{l} |\mu_{2lt-1}^{(n)}(x)+\mu_{2l(t-1)+2p-2}^{(n)}(x)| = O(1) \quad \text{... (9)}$$

$$\sum_{t=1}^{s}\sum_{p=1}^{l} |\mu_{2lt-1}^{(n)}(x)-\mu_{2l(t-1)+2p-1}^{(n)}(x)| = O(1) \quad \text{... (10)}$$

where O is independent of x, n.

PROOF : We only prove the formula (10), the (8) and (9) is similar to be obtained. Let us denote $i = 2lt - 1$, $j = 2l(t-1) + 2p - 1$, then i and j is odd integer, and $0 \le i-j \le 2(l-p) \le 2(l-1)$. By using (2) we have

$$\mu_i^{(n)}(x)-\mu_j^{(n)}(x) = \frac{1}{2n}\left[\sin n(x-x_i^{(n)})\, ctg\,\frac{1}{2}(x-x_i^{(n)}) - \sin n(x-x_i^{(n)}-(i-j)h)\, ctg\,\frac{1}{2}(x-x_i^{(n)}-(i-j)h)\right]. \quad \text{... (11)}$$

Let us denote

$$G(t) = \frac{\sin n(x-t)}{tg\,\frac{1}{2}(x-t)} - \frac{\sin n(x-t-(i-j)h)}{tg\,\frac{1}{2}(x-t-(i-j)h)} \quad \text{... (12)}$$

then

$$I = \int_0^{2\pi} |G(t)|\,dt \le 4(2l-1)\pi^2. \quad \text{... (13)}$$

The proof of (13). Using periodicity of the integrand and let $t_1 = x - t$, the limit of the integral is changeless, then

$$I = \int_0^{2\pi}\left|\frac{\sin nt_1}{tg\,\frac{1}{2}t} - \frac{\sin n(t_1-(i-j)h)}{tg\,\frac{1}{2}(t_1-(i-j)h)}\right| dt_1.$$

Using the property of even function, we have

$$I = 2\int_0^{\pi}\left|\frac{\sin nt_1}{tg\,\frac{1}{2}t} - \frac{\sin n(t_1-(i-j)h)}{tg\,\frac{1}{2}(t_1-(i-j)h)}\right| dt_1.$$

Let $N = 2n$, $\delta = \frac{(i-j)h}{4}$ and $t_2 = \frac{1}{2}t_1 - \delta$, then

$$I = 4\int_0^{\pi/2}\left|\frac{\sin N(t_2-\delta)}{tg\,(t_2-\delta)} - \frac{\sin N(t_2+\delta)}{tg\,(t_2+\delta)}\right| dt_2$$

On using $\sin N(t_2+\delta)=\sin N(t_2-\delta)$ and

$$\frac{2}{\pi}x \le \sin x \le x, \quad 0 \le x \le \frac{\pi}{2} \qquad \text{... (14)}$$

we obtain $$I \le 8\delta \int_0^{\pi/2} \left| \frac{\sin N(t_2-\delta)}{\sin(t_2-\delta)\sin(t_2+\delta)} \right| dt_2$$

$$= 8\delta \left\{ \int_0^{2\delta} + \int_{2\delta}^{\pi/2-\delta} + \int_{\pi/2-\delta}^{\pi/2} \right\} \triangleq I_1 + I_2 + I_3$$

to estimate I_1, when $0 \le t_2 \le 2\delta$, using

$$\left| \frac{\sin N(t_2-\delta)}{\sin(t_2-\delta)} \right| \le N$$

$$\sin(t_2+\delta) \ge \sin\delta \ge \frac{2}{\pi}\delta$$

we have $I_1 \le 8(l-1)\pi^2$. On using (14) we get

$$I_2 \le 2\delta\pi^2 \int_{2\delta}^{\pi/2-\delta} \frac{1}{t_2^2-\delta^2}\, dt_2 = \pi^2 \left(\ln \frac{t_2-\delta}{t_2+\delta} \right) \Bigg|_{2\delta}^{\pi/2-\delta} \le 3\pi^3$$

when $t_2 \in \left[\frac{\pi}{2}-\delta, \frac{\pi}{2} \right]$, and $n \gg 0$, we have $\sin(t_2 \pm \delta) \ge \frac{1}{2}$, thus we have $I_3 < \pi^2$. Combining I_1, I_2 and I_3 the formula (13) is valid. Now we prove (10), on using (11) and (12) we have

$$\sum_{t=1}^{s} \sum_{p=1}^{l} \left| \mu_{2lt-1}^{(n)}(x) - \mu_{2l(t-1)+2p-1}^{(n)}(x) \right| = \frac{1}{2n} \sum_{p=1}^{l-1} \sum_{t=1}^{s} |G(x_i)|. \qquad \text{... (15)}$$

Let $z_0 = 0$, $z_k = x_{2lk-1}^{(n)}$, $k = 1, 2, \ldots, s$, $z_{s+1} = 2\pi$, then

$$\sum_{t=1}^{s} |G(x_i)| = \sum_{k=1}^{s} |G(z_k)| = \frac{1}{z_{k+1}-z_k} \sum_{k=0}^{s} \int_{z_k}^{z_{k+1}} |G(z_k)|\, dt$$

$$\le \frac{1}{z_{k+1}-z_{k-1}} \sum_{k=0}^{s} \int_{z_k}^{z_{k+1}} \big| |G(z_k)| - |G(t)| \big|\, dt$$

$$+ \frac{1}{z_{k+1}-z_k} \sum_{k=0}^{s} \int_{z_k}^{z_{k+1}} |G(t)|\, dt.$$

On using (13) and $\big| |a| - |b| \big| \le |a-b|$ we get

$$\sum_{i=1}^{s} |G(x_i)| \le \frac{1}{z_{k+1}-z_k} \sum_{k=0}^{s} \int_{z_k}^{z_{k+1}} \left(\int_{z_k}^{z_{k+1}} |G'(t)|\,dt \right) dt + 8n(2l-1)\pi$$

$$= \sum_{k=0}^{s} \int_{z_k}^{z_{k+1}} |G'(t)|\,dt + 8n(2l-1)\pi.$$

On using Bernstein's inequality we have

$$\sum_{i=1}^{s} |G(x_i)| \le n \cdot \max_x \sum_{k=0}^{s} \int_{z_k}^{z_{k+1}} |G(t)|\,dt + 8n(2l-1)\pi$$

$$\le 4n(2l-1)(\pi^2 + 2\pi)$$

therefore, by (15) we have

$$\frac{1}{2n} \sum_{p=1}^{l-1} \sum_{i=1}^{s} |G(x_i)| \le (2l-1)^2 (\pi^2 + 2\pi).$$

The formula (10) is proved. The lemma is proved.

3. Proof of the Theorem

We know that Theorem 1 is valid by Theorem 2.

Proof of the Theorem 2 — From (3), (5) and (6) we have

$$H_n(f; r, x) - f(x)$$

$$= \left\{ \frac{1}{2^{r+1}} \sum_{k=0}^{2n} \sigma_h^{r+1} f(x_k^{(n)})\, \mu_k^{(n)}(x) + \sum_{k=0}^{2n} f(x_k^{(n)})\, \mu_k^{(n)}(x) - f(x) \right\}$$

$$+ \sum_{t=1}^{s} \sum_{p=1}^{l} \frac{1}{2^{r+1}} \sigma_h^{r+1} f(x_{2l(t-1)+2p-1}^{(n)})\,(\mu_{2lt-1}^{(n)}(x) - \mu_{2l(t-1)+2p-1}^{(n)}(x))$$

$$- \sum_{t=1}^{s} \sum_{p=1}^{l} \frac{1}{2^{r+1}} \sigma_h^{r+1} f(x_{2l(t-1)+2p-2}^{(n)})\,(\mu_{2lt-1}^{(n)}(x) + \mu_{2l(t-1)+2p-2}^{(n)}(x))$$

$$- \sum_{p=2ls}^{2n} \frac{1}{2^{r+1}} \sigma_h^{r+1} f(x_p^{(n)})\, \mu_p^{(n)}(x)$$

$$\triangleq \sum^{4} e_j \,. \qquad \text{... (16)}$$

Let $p(x) \in H_n^T$ and has minimum deviation with the function $f(x)$. Then

$$|p(x) - f(x)| \le E_n^*(f). \qquad \text{... (17)}$$

Because $p(x)$ is equal to its interpolation trigonometric polynomial, we have

$$\sum_{k=0}^{2n} \left(\frac{1}{2^{r+1}} \sigma_h^{r+1} p(x_k^{(n)}) - p(x_k^{(n)}) \right) \mu_k^{(n)}(x) = \frac{1}{2^{r+1}} \sigma_h^{r+1} p(x) + p(x) \qquad \text{... (18)}$$

thus

$$e_1 = \sum_{k=0}^{2n} \left\{ \frac{1}{2^{r+1}} \sigma_h^{r+1} (f(x_k^{(n)}) - p(x_k^{(n)})) + (f(x_k^{(n)}) - p(x_k^{(n)})) \right\} \mu_k^{(n)}(x)$$

$$+ \left[\frac{1}{2^{r+1}} \sigma_h^{r+1} (p(x) - f(x)) + p(x) - f(x) \right] + \frac{1}{2^{r+1}} \sigma_h^{r+1} f(x)$$

$$\triangleq d_1 + d_2 + d_3.$$

From (17), we have $|d_2| \le E_n^*(f)$. On using the the relation of jth derivative and difference of jth order, when $f(x) \in C_{2\pi}^j$, then

$$\sigma_h^j f(x) = \left(-\frac{\pi}{n} \right)^j f^{(j)}(\xi_x) \qquad \text{... (19)}$$

where ξ_x lies between $x - h$ and $x + (j - 1)h$. By (19) we obtain

$$d_3 = \frac{1}{2^{r+1}} \sum_{i=0}^{r+1-j} (-1)^i \binom{r+1-j}{i} \sigma_h^j f(x + ih)$$

$$= \frac{1}{2^{r+1}} \left(-\frac{\pi}{n} \right)^j \sum_{i=0}^{r+1-j} (-1)^i \binom{r+1-j}{i} f^{(j)}(\xi_{x+ih})$$

$$= O\left(\frac{1}{n^j} \omega\left(f^j, \frac{1}{n} \right) \right). \qquad \text{... (20)}$$

To estimate d_1, on using (17) and (8) we have

$$|d_1| = \left| \sum_{k=0}^{2n} (f(x_k^{(n)}) - p(x_k^{(n)})) \frac{1}{2^{r+1}} \sum_{i=0}^{r+1} \binom{r+1}{i} \right.$$

$$\left. \times \; (\mu_k^{(n)}(x) + (-1)^i \mu_{k+1-i}^{(n)}(x)) \right|$$

$$= O(E_n^*(f)).$$

Combining d_1, d_2 and d_3, we have

$$|e_1| = O\left(E_n^*(f) + \frac{1}{n^j}\,\omega\left(f^j, \frac{1}{n}\right)\right).$$

On using (9), (10), (20) and

$$\mu_k^{(n)}(x) = O(1), \quad k = 0, 1, \ldots, 2n \qquad \ldots (21)$$

we get

$$e_\nu = O\left(\frac{1}{n^j}\,\omega\left(f^j, \frac{1}{n}\right)\right), \quad \nu = 2, 3, 4.$$

Therefore, combining e_ν ($\nu = 1, \ldots, 4$), Theorem 2 is proved.

Remark : The highest convergence order of $H_n(f; r, x)$ does not exceed $\frac{1}{n^{r+2}}$. In fact, by the interpolation property of $H_n(f; r, x)$ we have

$$H_n(f; r, x_{2lt-1}^{(n)}) - f(x_{2lt-1}^{(n)}) = B_{2lt-1}.$$

Let $f_0(x) = \cos x$ and $l = 1$. Then

$$B_{2lt-1} = \frac{1}{2^{r+1}}\,\sigma_h^{r+2} f_0(x_{2l(t-1)}^{(n)}) = \frac{1}{2^{r+1}}\left(-\frac{\pi}{n}\right)^{r+2} f_0^{(r+2)}(\xi)$$

where ξ lies between $x_{2l(t-1)}^{(n)} - h$ and $x_{2l(t-1)}^{(n)} + (r+2)h$. If $n >> 0$ and $\tilde{t} = \left[\frac{s}{8}\right]$, then

$$|B_{2l\tilde{t}-1}| \approx \frac{\sqrt{2}}{2^{r+2}}\left(\frac{\pi}{n}\right)^{r+2}.$$

REFERENCES

1. И. П. Натансон, Конструктивная Теория функций, Государственное издательство технико-теоретической литературы, Москва 1949 Ленинград
2. S. N. Bernstein, Sur une modification de la formula d'interpolation de Lagrange. Xapbkōb, ЗаП. Матем. т-ва (4) 5 (1932), 49-57
3. Laiyi Zhu, *Acta Math. Sinica* No. 1 (1993), 136-44.