Indian J. pure appl. Math., **27**(1) : 73-84, January 1996

GENERALIZED SEQUENCE SPACE $c_0 (X, \lambda, p)$

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(*Received* 14 *October* 1993; *after revision* 19 *April* 1995; *accepted* 30 *August* 1995)

In this paper we study $c_0(X, \lambda, p)$ as a generalization of $c_0(p)$. We characterize Köthe-Toeplitz duals of $c_0(X, \lambda, p)$ and $c_0(B(X, Y), \lambda, p)$ and continuous dual of $c_0(X, \lambda, p)$.

1. INTRODUCTION

Let $p = (p_k)$ and $q = (q_k)$ be any sequences of strictly positive real numbers and $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ be any sequences of non-zero complex numbers. Let X and Y be the Banach spaces over the field C of complex numbers and $B(X, Y)$ be the Banach space of all bounded linear operators from X into Y with the usual operator norm. If $T \in B(X, Y)$ the operator norm of T is $\| T \| = \sup \{\| Tx \| : x \in S\}$ where $S = \{x \in X : \|x\| \leq 1\}$. X^* will denote the continuous dual of X. The zero element of X, Y, $B(X, Y)$ and X^* will be denoted by θ. Maddox[6] introduced $c_0(p)$, which is a generalization of the well known sequence space c_0 (for further details see Maddox[7,8] and Lascarides[4]). We introduce the following set of X-valued sequences

$$c_0(X, \lambda, p) = \{\bar{x} = (x_k) : x_k \in X, \ \| \lambda_k x_k \|^{p_k} \to 0 \quad \text{as} \quad k \to \infty\}.$$

Similarly, we define

$$c_0(B(X, Y), \lambda, p) = \{\bar{A} = (A_k) : A_k \in B(X, Y), \ \| \lambda_k A_k \|^{p_k} \to 0 \text{ as } k \to \infty\},$$

Further $c_0(X, \lambda, p)$ will be denoted by $c_0(X, p)$ when $\lambda_k = 1$ for all $k \geq 1$, and by $c_0(X, \lambda)$ when $p_k = 1$ for all $k \geq 1$, (Rath[10] has denoted $c_0(X, p)$ by $c_0(p, X)$).

The generalized Köthe-Toeplitz duals, the α- and β- duals, for the set $E(X)$ of X-valued sequences have been defined as :

Definition 1.1 (Maddox[9]) — Let X and Y be Banach spaces and $\bar{A} = (A_k)$ a sequence of linear, but not necessarily bounded, operators A_k on X into Y. Suppose $E(X)$ is a non-empty set of X-valued sequences. Then the α-dual of $E(X)$ is defined by

$$E^{\alpha}(X) = \{\bar{A} = (A_k) : \sum_{k=1}^{\infty} \|A_k x_k\| \text{ converges for all } \bar{x} \in E(X)\}$$

and β-dual of $E(X)$ is defined as

$$E^{\beta}(X) = \{\bar{A} = (A_k) : \sum_{k=1}^{\infty} A_k x_k \text{ converges for all } \bar{x} \in E(X)\}$$

where the convergence in $E^{\beta}(X)$ is with respect to the norm of Y.

Robinson[11] considered the action of infinite matrices of linear operators from a Banach space into another on sequences of elements of the first Banach space. The α- and β-duals for various vector valued sequence spaces have been obtained in terms of sequences of operators (see Maddox[9]).

Srivastava and Ansari[12] have introduced Köthe-Toeplitz duals for the set $E(B(X, Y))$ of $B(X, Y)$-valued sequences which are defined as follows :

Definition 1.2 (Srivastava and Ansari[12], p.101) — Let X and Y be Banach spaces and $E(B(X, Y))$ be a non-empty set of $B(X, Y)$-valued sequences, then α- and β-duals are respectively defined by

$$E^{\alpha}(B(X, Y)) = \{\bar{x} = (x_k) : x_k \in X, \sum_{k=1}^{\infty} \|A_k x_k\| \text{ converges for all } \bar{A} \in E(B(X, Y))\},$$

and

$$E^{\beta}(B(X, Y)) = \{\bar{x} = (x_k) : x_k \in X, \sum_{k=1}^{\infty} A_k x_k \text{ converges in } Y \text{ for all } \bar{A} \in E(B(X, Y))\}.$$

The Köthe-Toeplitz duals of various sets of $B(X, Y)$-valued sequences have been obtained in Srivastava and Ansari[12, 13].

In this note we propose to study linear topological structure of $c_0(X, \lambda, p)$ endowed with natural paranorm and characterize operator version of Köthe-Toeplitz duals of and continuous duals of $c_0(X, \lambda, p)$. We also investigate Köthe-Toeplitz duals of $c_0(B(X, Y), \lambda, p)$.

$c_0(X, \lambda, p)$ is the generalization of several known sequence spaces, for instance the following classes arise from $c_0(X, \lambda, p)$ as the special cases :

(i) $X = \mathbf{C}$, $\lambda_k = 1$, for all k, then $c_0(X, \lambda, p) = c_0(p)$ (Maddox[6]).

(ii) $X = \mathbf{C}$, $\lambda_k = 1$ and $p_k = \frac{1}{k}$, for all k, then $c_0(X, \lambda, p) = \Gamma$ (Iyer[1]),

where Γ is the space of all complex sequences $\bar{x} = (x_k)$ such that $|x_k|^{1/k} \to 0$ as $k \to \infty$. Γ can be regarded as the space of all integral functions $f(z) = \sum_{k=1}^{\infty} x_k z^k$.

(iii) $X = \mathbf{C}$, $p_k = \frac{1}{k}$, for all k, then $c_0(X, \lambda, p) \cap \Gamma = \Gamma(\lambda)$, (Titus[15]),

where $\Gamma(\lambda)$ is the subspace of Γ consisting of complex sequences $\bar{x} = (x_k) \in \Gamma$ such that $\lambda \circ \bar{x} = (\lambda_k x_k) \in \Gamma$.

(iv) $X = \mathbf{C}$, $\lambda_k = k!$ and $p_k = \frac{1}{k}$, for all k, then $c_0(X, \lambda, p) = \chi$, (Kamthan[2]), where the sequence space χ consists of all complex sequences $\bar{x} = (x_k)$ such that $(k\,!\,|x_k|)^{1/k} \to 0$ as $k \to \infty$ and χ can be regarded as a collection of all integral functions $f(z) = \sum_{k=1}^{\infty} x_k z^k$ of exponential order 1 and type 0.

(v) $X = \mathbf{C}$, $(|\lambda_k|)$ is non-decreasing and $\liminf_k |\lambda_k|^{-p_k} = r$, $0 \le r < \infty$ then $c_0(X, \lambda, p) = \hat{D}_0(p)$, (Srivastava and Ratha[14]).

(vi) $\lambda_k = 1$ and $p_k = \frac{1}{k}$, for all k, then $c_0(X, \lambda, p) = E_0(X)$ (Srivastava and Ansari[13]) where $E_0(X) = \{\bar{x} = (x_k) : x_k \in X$, and $\|x_k\|^{1/k} \to 0$ as $k \to \infty\}$ (see also Lalitha[3], who has considered $E_0(X)$ as the space of integral functions $f(z) = \sum_{k=1}^{\infty} x_k z^k$ with coefficients from a Banach space X).

(vii) $\lambda_k = 1$ and $p_k = 1$, for all k, then $c_0(X, \lambda, p) = c_0(X)$ (Maddox[9]).

(viii) $\lambda_k = 1$, for all k, then $c_0(X, \lambda, p) = c_0(p, X)$, (Rath[10]).

2. Paranormed Space $c_0(X, \lambda, p)$

In this section we study linear topological structure of $c_0(X, \lambda, p)$. We denote $r_k = p_k^{-1}$, $s_k = q_k^{-1}$ and $t_k = \left|\frac{\lambda_k}{\mu_k}\right|^{p_k}$.

Lemma 2.1 — For any $p = (p_k)$, $c_0(X, \lambda, p) \subset c_0(X, \mu, p)$ if and only if

$$\liminf_k t_k > 0. \qquad \text{... (2.1)}$$

PROOF : Sufficiency of (2.1) is straightforward. Suppose $\liminf_k t_k = 0$ then there exists a subsequence $(k_{(n)})$ of (k) such that for each $n \ge 1$

$$n\,|\lambda_{k(n)}|^{p_{k(n)}} < |\mu_{k(n)}|^{p_{k(n)}}.$$

Now the sequence $\bar{x} = (x_k)$ defined by $x_{k(n)} = \lambda_{k(n)}^{-1} n^{-r_{k(n)}} z$, $n \ge 1$ and $x_k = \theta$, otherwise, where $z \in X$ and $\|z\| = 1$ is in $c_0(X, \lambda, p)$ but $\|\mu_{k(n)} x_{k(n)}\|^{p_{k(n)}} > 1$, for each $n \ge 1$, implies that $\bar{x} \notin c_0(X, \mu, p)$ and hence the necessity of (2.1) follows. This completes the proof.

Lemma 2.2 — For any $p = (p_k)$, $c_0(X, \mu, p) \subset c_0(X, \lambda, p)$ if and only if

$$\limsup_k t_k < \infty. \quad \text{... (2.2)}$$

PROOF : Let (2.2) hold. Then there exists $L > 0$ such that $|\lambda_k|^{p_k} < L|\mu_k|^{p_k}$ for all large values of k, which implies that $c_0(X, \mu, p) \subset c_0(X, \lambda, p)$.

On the other hand if $\limsup_k t_k = \infty$, then there exists a subsequence $(k_{(n)})$ of (k) such that for each $n \geq 1$, $|\lambda_{k(n)}|^{p_{k(n)}} > n|\mu_{k(n)}|^{p_{k(n)}}$. Thus the sequence $\bar{x} = (x_k)$, where for $z \in X$ with $\| z \| = 1$, $x_{k(n)} = \mu_{k(n)}^{-1} n^{-r_{k(n)}} z$, $n \geq 1$ and $x_k = \theta$, otherwise, is in $c_0(X, \mu, p)$ but $\bar{x} \notin c_0(X, \lambda, p)$. This completes the proof.

If we combine the Lemmas 2.1 and 2.2, we immdeiately get the following theorem :

Theorem 2.3 — For any $p = (p_k)$, $c_0(X, \lambda, p) = c_0(X, \mu, p)$ if and only if

$$0 < \liminf_k t_k \leq \limsup_k t_k < \infty.$$

Corollary 2.4 — For any $p = (p_k)$,

(i) $c_0(X, \lambda, p) \subset c_0(X, p)$ if and only if $\liminf_k |\lambda_k|^{p_k} > 0$;

(ii) $c_0(X, p) \subset c_0(X, \lambda, p)$ if and only if $\limsup_k |\lambda_k|^{p_k} < \infty$;

(iii) $c_0(X, \lambda, p) = c_0(X, p)$ if and only if

$$0 < \liminf_k |\lambda_k|^{p_k} \leq \limsup_k |\lambda_k|^{p_k} < \infty.$$

In view of Lemma 1 of Maddox[6], the following Lemmas 2.5 and 2.6 can easily be proved.

Lemma 2.5 — For any $\lambda = (\lambda_k)$, $c_0(X, \lambda, p) \subset c_0(X, \lambda, q)$ if and only if

$$\liminf_k \frac{q_k}{p_k} > 0. \quad \text{... (2.3)}$$

Lemma 2.6 — For any $\lambda = (\lambda_k)$, $c_0(X, \lambda, q) \subset c_0(X, \lambda, p)$ if and only if

$$\limsup_k \frac{q_k}{p_k} < \infty. \quad \text{... (2.4)}$$

Combining Lemmas 2.5 and 2.6 we get the following theorem :

Theorem 2.7 — For any $\lambda = (\lambda_k)$, $c_0(X, \lambda, p) \subset c_0(X, \lambda, q)$ if and only if

$$0 < \liminf_k \frac{q_k}{p_k} \leq \limsup_k \frac{q_k}{p_k} < \infty.$$

Corollary 2.8 — For any $\lambda = (\lambda_k)$,

(i) $c_0(X, \lambda) \subset c_0(X, \lambda, p)$ if and only if $\liminf_k p_k > 0$;

(ii) $c_0(X, \lambda, p) \subset c_0(X, \lambda)$ if and only if $\limsup_k p_k < \infty$;

(iii) $c_0(X, \lambda, p) = c_0(X, \lambda)$ if and only if

$$0 < \liminf_k p_k \leq \limsup_k p_k < \infty.$$

Corollary 2.9 — For any sequences $\lambda = (\lambda_k)$, $\mu = (\mu_k)$, $p = (p_k)$ and $q = (q_k)$, $c_0(X, \lambda, p) \subset c_0(X, \mu, q)$ if and only if (2.1) and (2.3) hold.

PROOF : Proof follows easily from Lemmas 2.1 and 2.5.

In the following example we show that $c_0(X, \lambda, p)$ may strictly be contained in $c_0(X, \mu, q)$ while (2.1) and (2.3) hold.

Example 2.10 — Let $\bar{x} = (x_k)$ be a sequence in the Banach space X, such that $\| x_k \| = k^{-k}$. Take $p_k = k^{-1}$ if k is odd, $p_k = k^{-2}$ if k is even, $q_k = k^{-1}$ for all k, $\lambda_k = 3^k$, for all k and $\mu_k = 2^k$ for all k. Clearly (2.1) and (2.3) hold, and $\bar{x} \in c_0(X, \mu, q)$ but $\bar{x} \notin c_0(X, \lambda, p)$.

We take vector operations co-ordinatewise and we see that $p = (p_k) \in l_\infty$ is necessary and sufficient condition for the linearity of $c_0(X, \lambda, p)$ (see Lascarides[4], p. 497). The zero vector of $c_0(X, \lambda, p)$ will be denoted by $\bar{\theta} = (\theta, \theta, \theta, ...)$. For $\bar{x} \in c_0(X, \lambda, p)$, we define

$$P_{\lambda, p}(\bar{x}) = \sup_k \| \lambda_k x_k \|^{p_k/M} \qquad \text{... (2.5)}$$

where $\quad M = \max(1, \sup_k p_k)$.

Theorem 2.11 — Let $p = (p_k) \in l_\infty$. If X is a normed space then $c_0(X, \lambda, p)$ is a (total) paranormed space. Further if X is a Banach space then $c_0(X, \lambda, p)$ is an *FK*-space.

PROOF : We remark here that terms the paranorm, the metric and the topology are in connection with $P_{\lambda, p}$ defined by (2.5) and throughout the theorem p will $p_{\lambda, p}$.

We can easily prove that (i) if $\bar{x}^{(n)} \to \bar{\theta}$ in P and $\alpha_n \to \alpha$ then $\alpha_n \bar{x}^{(n)} \to \bar{\theta}$ in P and (ii) if $\alpha_n \to 0$ and $\bar{x} \in c_0(X, \lambda, p)$ then $\alpha_n \bar{x} \to \bar{\theta}$ in P, hence follows the continuity of scalar multiplication of P (see Wilansky[16]), and all other conditions for P to be a total paranorm are straightforward.

Let X be a Banach space. Suppose $(\bar{x}^{(n)})$ is a Cauchy sequence in $c_0(X, \lambda, p)$. Then due to completeness of X we get a sequence $\bar{x} = (x_k)$, $(x_k \in X)$, such that for each k, $x_k^{(n)} \to x_k$, as $n \to \infty$ and $P(\bar{x}^{(n)} - \bar{x}) \to 0$ as $n \to \infty$. Moreover $\bar{x} \in c_0(X, \lambda, p)$ follows from

$$\| \lambda_k x_k \|^{p_k/M} \leq P(\bar{x}^{(n)} - \bar{x}) + \| \lambda_k x_k^{(n)} \|^{p_k/M}$$

when n is taken sufficiently large.

For $k \geq 1$ the continuity of $P_k : c_0(X, \lambda, p) \to X$, where $P_k(\bar{x}) = x_k$, follows from $\| P_k(\bar{x}) \| \leq |\lambda_k|^{-1} [P(\bar{x})]^{M/p_k}$. Thus $c_0(X, \lambda, p)$ is an *FK*-space. This completes the proof.

Theorem 2.12 — Let $p = (p_k)$, $q = (q_k) \in l_\infty$.

(i) Let (2.1) hold. The identity mapping of $(c_0(X, \lambda, p), P_{\lambda, p})$ into $(c_0(X, \mu, p), P_{\mu, p})$ is continuous.

(ii) Let (2.3) hold. The identity mapping of $(c_0(X, \lambda, p),\ P_{\lambda, p})$ into $(c_0(X, \lambda, q),\ P_{\lambda, q})$ is continuous.

(iii) Let (2.1) and (2.3) hold. The identity mapping of $(c_0(X, \lambda, p),\ P_{\lambda, p})$ into $(c_0(X, \mu, q),\ P_{\mu, q})$ is continuous.

PROOF : Proof easily follows from Theorem 2.11 and Corollary 3 of Wilansky[17] (p.56).

3. KÖTHE-TOEPLITZ DUALS

Theorem 3.1 — For any $p = (p_k)$, $\bar{A} = (A_k) \in c_0^{\alpha}(X, \lambda, p)$ if and only if

(i) there exists an integer $m \geq 1$ such that $A_k \in B(X, Y)$ for each $k \geq m$, and

(ii) there exists an integer $N > 1$ such that $\sum_{k=m}^{\infty} |\lambda_k|^{-1} \|A_k\| N^{-r_k} < \infty$.

PROOF : Suppose that (i) and (ii) hold and $\bar{x} \in c_0(X, \lambda, p)$. Choose $K \geq m$ such that $\|\lambda_k x_k\|^{p_k} < \frac{1}{N}$ for all $k \geq K$. Then

$$\sum_{k=K}^{\infty} \|A_k x_k\| \leq \sum_{k=K}^{\infty} |\lambda_k|^{-1} \|A_k\| N^{-r_k} < \infty$$

implies that $\bar{A} \in c_0^{\alpha}(X, \lambda, p)$.

Conversely, suppose that $\bar{A} \in c_0^{\alpha}(X, \lambda, p)$. If (i) fails then there exists a subsequence $(k(i))$ of (k) such that $A_{k(i)} \notin B(X, Y)$ for each $i \geq 1$. Thus for each $i \geq 1$ we can find $z_{k(i)} \in S$ such that $\|A_{k(i)} z_{k(i)}\| > |\lambda_{k(i)}| i^{r_{k(i)}}$. Now the sequence $\bar{x}$ defined by

$$x_k = \begin{cases} \lambda_{k(i)}^{-1} i^{-r_{k(i)}} z_{k(i)}, & \text{for } k = k(i),\ i \geq 1, \text{ and} \\ \theta, & \text{otherwise,} \end{cases}$$

is in $c_0(X, \lambda, p)$ but $\|A_{k(i)} x_{k(i)}\| > 1$ for each $i \geq 1$ and so $\sum_{k=1}^{\infty} \|A_k x_k\| = \infty$, which contradicts that $\bar{A} \in c_0^{\alpha}(X, \lambda, p)$.

Next suppose that (ii) fails, i.e. $\sum_{k=m}^{\infty} |\lambda_k|^{-1} \|A_k\| N^{-r_k} = \infty$ for each $N > 1$. Now choose $m = k(1) < k(2) < k(3)$ .... such that $\sum_{s(N)} |\lambda_k|^{-1} \|A_k\| N^{-r_k} > 2$ where $S(N) = \{k(N-1), k(N-1)+1, \ldots, k(N)-1\}$, $N > 1$. Moreover for each $k \geq m$ there exists $z_k \in S$ such that $\|A_k\| < 2\|A_k z_k\|$. Now define $\bar{x}$ by

$$x_k = \begin{cases} \lambda_k^{-1} N^{-r_k} z_k, & \text{if } k \in S(N),\ N > 1, \\ \theta, & \text{otherwise.} \end{cases}$$

Then $\bar{x} \in c_0(X, \lambda, p)$ but

$$\sum_{s(N)} |\lambda_k|^{-1} \|A_k z_k\| N^{-r_k} \geq \frac{1}{2} \sum_{s(N)} |\lambda_k|^{-1} \|A_k\| N^{-r_k} > 1$$

shows that $\sum_{k=m}^{\infty} \|A_k x_k\| = \infty$, which is again contradictory. This completes the proof.

If in the sequence $\bar{A} = (A_k)$ all $A_k \in B(X, Y)$ then the Theorem 3.1 can be re-stated as :

Theorem 3.1′ — For any $p = (p_k)$, $c_0^{\alpha}(X, \lambda, p) = M_0(B(X, Y), \lambda, p)$ where

$$M_0(B(X, Y), \lambda, p)$$

$$= \bigcup_{N>1} \left\{ \bar{A} = (A_k) : A_k \in B(X, Y),\ \sum_{k=1}^{\infty} |\lambda_k|^{-1} \|A_k\| N^{-r_k} < \infty \right\}. \quad \text{... (3.1)}$$

Let $(T_k) = (T_1, T_2, T_3, ...)$ be a sequence in $B(X, Y)$. Then the group norm of (T_k) is defined by

$$\|(T_k)\| = \sup \left\| \sum_{k=1}^{n} T_k x_k \right\| \quad \text{... (3.2)}$$

where the supremum is taken over all $n \geq 1$ and all $x_k \in S$. This concept was introduced by Robinson[11] and was termed as group norm by Lorentz and Macphail[5] (see also Maddox[9], p.5). Connected with the group norm we first prove a lemma required to characterize β-dual of $c_0(X, \lambda, p)$ in the Theorem 3.3 below.

Lemma 3.2 — Let (T_k) be a sequence in $B(X, Y)$ and denote $R_k = (T_k, T_{k+1}, T_{k+2}, ...)$. Then exactly one of the following is true :

(i) $\|R_k\| = \infty$ for all $k \geq 1$

(ii) $\|R_k\| < \infty$ for all $k \geq 1$.

PROOF : Suppose (i) does not hold. Then there exists some m such that $\|R_m\| < \infty$. Clearly from the inequality $\|R_k\| \geq \|R_{k+1}\|$ for all $k \geq 1$ (see Maddox[9], Proposition 2.3(ii) (b), pp.5-6) we get that $\|R_k\| < \infty$ for all $k \geq m$. Now

$$\left\| \sum_{k=m-1}^{n} T_k x_k \right\| \leq \left\| \sum_{k=m}^{n} T_k x_k \right\| + \|T_{m-1} x_{m-1}\|, \quad x_k \in S,$$

implies that $\|R_{m-1}\| \leq \|R_m\| + \|T_{m-1}\|$ and so $\|R_{m-1}\| < \infty$ as $T_{m-1} \in B(X, Y)$. Hence repeating the same arguments we get that $\|R_k\| < \infty$ for all $k < m$ and it completes the proof.

Theorem 3.3 — Let $p = (p_k) \in l_\infty$. Then $\bar{A} = (A_k) \in c_0^\beta(X, \lambda, p)$ if and only if

(i) there exists an integer $m \geq 1$ such that $A_k \in B(X, Y)$ for each $k \geq m$, and

(ii) $\| R_m(\lambda, N) \| < \infty$ for some $N > 1$, where

$$R_m(\lambda, N) = (\lambda_m^{-1} N^{-r_m} A_m, \ \lambda_{m+1}^{-1} N^{-r_{m+1}} A_{m+1}, \ \lambda_{m+2}^{-1} N^{-r_{m+2}} A_{m+2}, \ldots\ldots).$$

PROOF : Suppose (i) and (ii) hold and $\| R_m(\lambda, N) \| = H < \infty$. Let $\bar{x} \in c_0(X, \lambda, p)$. For a given $\varepsilon > 0$, choose $0 < \eta < 1$ so that $\eta H < \varepsilon$. Then there exists $K \geq m$ such that

$$\| x_k \| < | \lambda_k |^{-1} N^{-r_k} \eta^{r_k M}$$

for all $k \geq K$, where as usual $M = \max(1, \sup_k p_k)$. For $n \geq K$, we have by Proposition 2.3(ii) of [Maddox[9], p.5] that

$$\left\| \sum_{k=n}^{n+i} A_k x_k \right\| = \left\| \sum_{k=n}^{n+i} \lambda_k^{-1} N^{-r_k} A_k (\lambda_k N^{r_k} x_k) \right\|$$

$$\leq \| R_m(\lambda, N) \| \max_{n \leq k \leq n+i} (| \lambda_k | N^{r_k} \| x_k \|) \leq H\eta < \varepsilon$$

moreover since Y is complete therefore $\sum_{k=1}^{\infty} A_k x_k$ is convergent in Y and hence $\bar{A} \in c_0^\beta(X, \lambda, p)$.

Conversely (i) can easily to established on the lines of Theorem 3.1. Now suppose that $\bar{A} \in c_0^\beta(X, \lambda, p)$ but $\| R_m(\lambda, N) \| = \infty$ for each $N > 1$. Then by Lemma 3.2, $\| R_n(\lambda, N) \| = \infty$, for all $N > 1$ and for all $n \geq m$. Thus there exist $m = k(1) < k(2) < k(3) \ldots.$ a subsequence of (k) and a sequence (z_k) in S such that

$$\left\| \sum_{s(N)} \lambda_k^{-1} N^{-r_k} A_k z_k \right\| > 1$$

for all $N > 1$ where $S(N)$, for $N > 1$, is as defined in Theorem 3.1. Now, the sequence $\bar{x} = (x_k)$ defined by

$$x_k = \begin{cases} \lambda_k^{-1} N^{-r_k} z_k, & k \in S(N), \ N > 1, \text{ and} \\ \theta, & \text{otherwise,} \end{cases}$$

belongs to $c_0(X, \lambda, p)$ but for each $N \geq 2$

$$\left\| \sum_{s(N)} A_k x_k \right\| = \left\| \sum_{s(N)} A_k (\lambda_k^{-1} N^{-r_k} z_k) \right\| > 1,$$

shows that $\sum_{k=1}^{\infty} A_k x_k$ does not converge in Y, which leads to a contradiction. This completes the proof.

Remark 3.4 : We note that Theorem 1 of Rath[10], Proposition 3.4 of Maddox[9] and Proposition 2.1 of Srivastava and Ansari[13] can easily be derived from the Theorems 3.1, 3.1′ and 3.3, by choosing X, λ, and p appropriately.

If we take $Y = \mathbf{C}$, i.e. $B(X, \mathbf{C}) = X^*$, the space of all bounded (continuous) linear functionals on X, then corresponding to (3.1), we have

$$M_0(X^*, \lambda, p) = \bigcup_{N>1} \left\{ \bar{f} = (f_k) ; f_k \in X^* \sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| N^{-r_k} < \infty \right\}.$$

Theorem 3.5 — For any $p = (p_k)$, $Y = \mathbf{C}$ and $f_k \in X^*$ for $k \geq 1$ then

$$c_0^{\alpha}(X, \lambda, p) = c_0^{\beta}(X, \lambda, p) = M_0(X^*, \lambda, p).$$

PROOF : By Theorem 3.1′ we at once have $c_0^{\alpha}(X, \lambda, p) = M_0(X^*, \lambda, p)$ and since $Y = \mathbf{C}$ is complete therefore $c_0^{\alpha}(X, \lambda, p) \subset c_0^{\beta}(X, \lambda, p)$. Now suppose that $\bar{f} \notin M_0(X^*, \lambda, p)$, then using the technique involved in Theorem 3.1 and taking $m = 1$ therein we get $\sum_{s(N)} |\lambda_k|^{-1} \|f_k\| N^{-r_k} > 2$ for all $N > 1$. Moreover for each $k \geq 1$ there exists $z_k \in S$ such that $\|f_k\| < 2 \, |f_k(z_k)|$. Thus the sequence $\bar{x}$ defined by

$$x_k = \operatorname{sgn}(f_k(z_k)) \, |\lambda_k|^{-1} N^{-r_k} z_k, \quad k \in S(N), \; N > 1,$$

is in $c_0(X, \lambda, p)$ but $\sum_{k=1}^{\infty} f_k(x_k) = \infty$ and so $\bar{f} \notin c_0^{\beta}(X, \lambda, p)$. Hence $c_0^{\beta}(X, \lambda, p) \subset M_0(X^*, \lambda, p)$ and it completes the proof.

In the following theorem we investigate the continuous dual of $c_0(X, \lambda, p)$.

Theorem 3.6 — If $p = (p_k) \in l_\infty$ then $c_0^*(X, \lambda, p)$, the continuous dual of $(c_0(X, \lambda, p), P_{\lambda, p})$ is isomorphic to $M_0(X^*, \lambda, p)$.

PROOF : Let $F \in c_0^*(X, \lambda, p)$ and $\bar{x} \in c_0(X, \lambda, p)$. For $n \geq 1$ take $\bar{s}^{(n)}$ where $s_k^{(n)} = x_k$, for $1 \leq k \leq n$, and θ for $k > n$. Then $\bar{s}^{(n)} \to \bar{x}$ with respect to $P = P_{\lambda, p}$. Define $\delta_k(x) = (\theta, \theta, ..., x, \theta, \theta, ...)$, x at the kth place, then $\bar{s}^n = \sum_{k=1}^{n} \delta_k(x_k)$. Thus

$$F(\bar{x}) = \lim_{n \to \infty} F(\bar{s}^{(n)}) = \lim_{n \to \infty} \sum_{k=1}^{n} F(\delta_k(x_k)) = \sum_{k=1}^{\infty} f_k(x_k)$$

where we write $F(\delta_k(x)) = f_k(x)$, $k \geq 1$. Clearly for each $k \geq 1$, f_k is a linear functional on X. Further if $x_i \to \theta$ in X as $i \to \infty$ then $\delta_k(x_i) \to \delta_k(\theta) = \bar{\theta}$ in $c_0(X, \lambda, p)$ with respect to P and so $F(\delta_k(x_i)) \to F(\bar{\theta}) = 0$, i.e., $f_k(x_i) \to 0$ as $i \to \infty$ whence $f_k \in X^*$, for all $k \geq 1$. Thus $\sum_{k=1}^{\infty} f_k(x_k)$ is convergent and $\bar{f} = (f_k)$ is a sequence in X^*. Since the sequence $\bar{f}$ in X^* is such that $\sum_{k=1}^{\infty} f_k(x_k)$ is convergent for every $\bar{x} \in c_0(X, \lambda, p)$ therefore by Theorem 3.5 $\bar{f} \in M_0(X^*, \lambda, p)$. Hence each $F \in c_0^*(X, \lambda, p)$ corresponds to an $\bar{f} \in M_0(X^*, \lambda, p)$.

On the other hand if $\bar{f} \in M_0(X^*, \lambda, p)$ then by Theorem 3.5 $\sum_{k=1}^{\infty} f_k(x_k)$ is convergent for every $\bar{x} \in c_0(X, \lambda, p)$. Now define F on $c_0(X, \lambda, p)$ by $F(\bar{x}) = \sum_{k=1}^{\infty} f_k(x_k)$. Clearly F is linear. For continuity of F, let $\bar{x}^{(n)} \to \bar{\theta}$ in $(c_0(X, \lambda, p), P)$. Now for $\varepsilon > 0$ choose $0 < \eta < 1$ such that $\eta \sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| N^{-r_k} < \varepsilon$. Then for $\eta N^{-1/M} > 0$ ($M = \max(1, \sup_k p_k)$) there exists n_0 such that for $n \geq n_0$

$$\sup_k \|\lambda_k x_k^{(n)}\|^{p_k/M} < \eta N^{-1/M}.$$

This implies that for all $n \geq n_0$

$$|F(\bar{x}^{(n)})| \leq \sum_{k=1}^{\infty} |f_k(x_k^{(n)})| \leq \sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| N^{-r_k} \eta^{M/p_k}$$

$$\leq \eta \sum_{k=1}^{\infty} |\lambda_k|^{-1} \|f_k\| N^{-r_k} < \varepsilon.$$

Hence $F(\bar{x}^{(n)}) \to 0$ as $n \to \infty$, i.e., $F \in c_0^*(X, \lambda, p)$, which shows that each $\bar{f} \in M_0(X^*, \lambda, p)$ corresponds to an $F \in c_0^*(X, \lambda, p)$.

Now $\phi(F) = \bar{f}$ clearly defines an isomorphism of $c_0^*(X, \lambda, p)$ onto $M_0(X^*, \lambda, p)$. This completes the proof.

Remark 3.7 : We note that Corollary 6.15 of Maddox[9] and the continuous dual of the space of entire functions with coefficients from a Banach space Lalitha[3] can easily be deduced from Theorem 3.6.

We now define $M_0(X, \lambda, p)$ corresponding to (3.1) wherein $B(X, Y)$ is replaced by X.

Theorem 3.8 — $c_0^{\alpha}(B(X, Y), \lambda, p) = c_0^{\beta}(B(X, Y), \lambda, p) = M_0(X, \lambda, p)$.

PROOF : $M_0(X, \lambda, p) \subset c_0^{\alpha}(B(X, Y), \lambda, p)$ is straightforward (cf. Theorem 3.1') and since Y is a Banach space therefore $c_0^{\alpha}(B(X, Y), \lambda, p) \subset c_0^{\beta}(B(X, Y), \lambda, p)$. Now suppose that $\bar{x} \in c_0^{\beta}(B(X, Y), \lambda, p)$ but $\bar{x} \notin M_0(X, \lambda, p)$. Then $\sum_{k=1}^{\infty} |\lambda_k|^{-1} \|x_k\| N^{-r_k} = \infty$ for each $N > 1$. Thus we get $S(N) = \{k(N-1), k(N-1)+1, \ldots k(N)-1\}$ for each $N > 1$ such that

$$\sum_{s(N)} |\lambda_k|^{-1} \|x_k\| N^{-r_k} > 1.$$

By application of Hahn-Banach Theorem for each $k \geq 1$ we get $f_k \in X^*$ with $\|f_k\| = 1$ and $f_k(x_k) = \|x_k\|$. Now we define $\bar{A}$ by

$$A_k(z) = |\lambda_k|^{-1} N^{-r_k} f_k(z)\, y, \quad k \in S(N),$$

for all $z \in X$, where $y \in Y$ is fixed and $\|y\| = 1$. Naturally for $N \geq 2$, $k \in S(N)$, $\|A_k\| = |\lambda_k|^{-1} N^{-r_k}$ and so $\bar{A} \in c_0(B(X, Y), \lambda, p)$ but

$$\left\| \sum_{s(N)} A_k x_k \right\| = \sum_{s(N)} |\lambda_k|^{-1} \|x_k\| N^{-r_k} > 1,$$

whence $\sum_{k=1}^{\infty} A_k x_k$ is not convergent in Y, which implies that $\bar{x} \notin c_0^{\beta}(B(X, Y), \lambda, p)$. This completes the proof.

Corollary 3.9 — For any $p = (p_k)$ and $Y = \mathbf{C}$,

$$c_0^{\alpha}(X^*, \lambda, p) = c_0^{\beta}(X^*, \lambda, p) = M_0(X, \lambda, p).$$

Further we note that Proposition 3.1 of Srivastava and Ansari[13] can easily be deduced from Theorem 3.8.

Finally we conclude this paper with the remark that since $c_0(X, p)$, $c_0(X, \lambda)$, $E_0(X)$, $c_0(X)$, $c_0(p)$, $c_0(\lambda)$, $\hat{D}_0(p)$, $\Gamma(\lambda)$, χ, and Γ etc. can be obtained from $c_0(X, \lambda, p)$ by choosing X, λ and p suitably, therefore, if for any of these spaces the characterizations concerning the Köthe-Toeplitz duals and continuous dual (provided it exists) which remain unmentioned herein can easily be obtained accordingly from the results of the section 3 by taking X, λ, and p appropriately. Similarly we can deal with $c_0(B(X, Y), \lambda, p)$.

ACKNOWLEDGEMENT

This paper was written while J. K. Srivastava held a project (No. F. 8-5/90 RBB-II, dated July 3, 1991) of the University Grants Commission and B. K. Srivastava worked in it as a Project Assistant. The support by Commission is very gratefully acknowledged.

The authors are thankful to the referee for bringing Rath's paper[10] to their notice and for other valuable suggestions which improved the presentation of the paper.

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