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RAYLEIGH WAVE SCATTERING AT THE EDGE OF PACK-ICE IN SHALLOW OCEANS

P. S. DESHWAL AND S. RATHI

Department of Mathematics, Maharishi Dayanand University, Rohtak 124 001

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A theoretical study of Rayleigh wave scattering at the edge of pack-ice in shallow oceans is made in the present paper. There is a thin uniform distribution of pack-ice in the surface of an oceanic layer overlying a solid halfspace. The Wiener-Hopf technique is the method of solution. The results are derived in terms of integrals whose evaluation along suitable contours gives the reflected, refracted, diffracted and surface waves appropriate to the impeding surface. The scattered waves propagate with the speeds of the waves in the solid and decay exponentially. Numerical computations are carried out for the amplitude of the scattered waves versus the wave number showing sharp fall of the amplitude with slight increase in the wave number.

1. INTRODUCTION

We aim to study scattering effects of seismic waves due to the presence of pack-ice in shallow oceans. The energy of a body wave is lessened because of its partial conversion into reflected body waves, surface waves and sacttered waves. In seismology, diffraction of body waves by a surface discontinuity is used in measuring conversion of body waves to surface waves propagating along the discontinuous surface (Boore[1]).

The effect of a vertical barrier, fixed in an infinitely deep sea, on normally incident surface waves, was considered by Ursell[2] for a two-dimensional case. Gregory[3] has studied the attenuation of Rayleigh waves due to the presence of a surface impedance in the surface of a solid half space. He has offered various interpretations to the surface impedance to suit the physical conditions and has used the technique of Wiener and Hopf[4] in solving his problem. Scattering of Rayleigh waves at the corner of an elastic quarter space has been discussed recently by Momoi[5] using the technique of Fourier transformation. He has used numerous approximations for evaluation of integral transforms to obtain expressions for the energies of Rayleigh waves along the two free surfaces. Mann and Deshwal[6] have used the Wiener-Hopf technique to discuss the scattering behaviour of Rayleigh waves due to a plane barrier in the surface of a shallow ocean.

The present problem represents the model of the oceanic crustal layer overlying the solid mantle of the Earth. The ice is supposed to be uniformly distributed on half of the surface of the liquid layer. The thin, uniform and smooth distribution of pack-ice is such that it exerts a normal stress proportional to the normal acceleration. If p_{zz} be the normal stress and (U, W) be the displacement components in the layer, then the pack-ice satisfies the condition

$$p_{zz} = a' d^2W/dt^2$$

where a' depends upon the physical properties of the ice and d^2W/dt^2 is the normal acceleration.

2. Statement of the Problem

The problem is two-dimensional. Wave propagation takes place in the zx-plane (Fig. 1). A liquid layer of thickness H lies over a solid halfspace. The media are homogeneous, isotropic and slightly dissipative. The origin of the coordinate system is taken on the liquid-solid interface with the x-axis in the interface and the z-axis pointing vertically downwards. The wave equation in the layer is

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \bar{\phi}}{\partial t^2} + \frac{\varepsilon}{c^2}\frac{\partial \bar{\phi}}{\partial t}, \qquad \text{... (1)}$$

where $\varepsilon > 0$ is the damping constant and c is the velocity of propagation. Let the potential be harmonic in time, i.e.

$$\bar{\phi}(x, z, t) = \phi(x, z)\, e^{-iwt}, \qquad \text{... (2)}$$

then eqn. (1) takes the form

$$(\nabla^2 + k^2)\phi = 0. \qquad \text{... (3)}$$

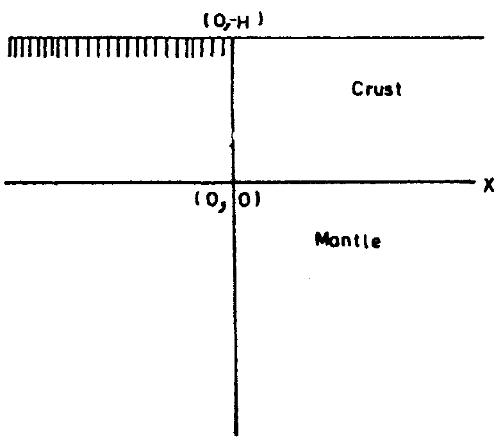

Fig. 1. Pack-ice in the surface of shallow ocean overlying a solid half-space.

The wave equations in solid are :

$$(\nabla^2 + k'^2)\phi_1 = 0 \qquad \text{... (4)}$$

$$(\nabla^2 + k''^2)\Psi_1 = 0 \qquad \text{... (5)}$$

where $$k = k_1 + ik_2, \quad k' = k_1' + ik_2', \quad k'' = k_1'' + ik_2''. \qquad \text{... (6)}$$

The imaginary parts of k's are assumed to be small and positive. ϕ, ϕ_1, Ψ_1 are the potentials in shallow ocean and solid halfspace respectively.

Let the incident wave in solid and liquid media propagating from the side $x > 0$, be

$$\phi_{1i} = \frac{D(2\alpha_N^2 - k''^2)\,\gamma_N \cos\gamma_N H \exp[-i\alpha_N x - \gamma_{1N} z]}{\gamma_{1N}(k'')^2}, \quad z \geq 0 \qquad \text{... (7)}$$

$$\Psi_{1i} = \frac{2iD\alpha_N \gamma_N \cos\gamma_N H \exp[-i\alpha_N x - \delta_{1N} z]}{(k'')^2}, \quad z \geq 0 \qquad \text{... (8)}$$

and $$\phi_i = D \sin\gamma_N (z + H) \exp(-i\alpha_N x), \quad -H \leq z \leq 0 \qquad \text{... (9)}$$

where α_N is a root of frequency equation

$$\tan\gamma_N H = \frac{[4\alpha_N^2 \gamma_{1N} \delta_{1N} - (2\alpha_N^2 - k''^2)^2]\,\rho_1 \gamma_N}{\rho (k'')^4 \gamma_{1N}} \qquad \text{... (10)}$$

ρ_1 and ρ are densities of the solid and the liquid and D is a constant. Further,

$$\gamma_N^2 = k^2 - \alpha_N^2, \quad \gamma_{1N}^2 = \alpha_N^2 - (k')^2, \quad \delta_{1N}^2 = \alpha_N^2 - (k'')^2. \qquad \text{... (11)}$$

Let the total potentials be

$$\phi_t = \phi + \phi_i, \quad \phi_{1t} = \phi_{1i} + \phi_1, \quad \Psi_{1t} = \Psi_{1i} + \Psi_1. \qquad \text{... (12)}$$

The subscripts t and i denote the total and incident potentials, and the subscript 1 denotes the quantities in the solid.

3. Boundary Conditions

(i) $\phi_1(x, z)$ and $\Psi_1(x, z)$ are bounded, when z is infinite, ... (13)

(ii) $p_{zz} = 0, \quad z = -H, \quad x > 0,$ ... (14)

$$= -a' w^2 \frac{\partial \phi_t}{\partial z}, \quad z = -H, \quad x < 0 \qquad \text{... (15)}$$

(iii) $W = W_1, \; p_{zz} = (p_{zz})_1, \; (p_{zx})_1 = 0, \quad z = 0$, for all x. ... (16)

The time factor exp $(-iwt)$ is understood in the displacement potential. The condition (15) implies that there is a uniform distribution of ice exerting a normal pressure proportional to the normal acceleration. It simplifies to

$$\phi_t(x, z) = \frac{-a\partial\phi_t}{\partial z}, \quad z = -H, \quad x < 0 \qquad \text{... (17)}$$

where p_{zz} and p_{zx} are normal and shearing stresses and a is a constant depending upon the material of the pack-ice. We define the Fourier transforms

$$\bar{\phi}(\alpha, z) = \int_{-\infty}^{\infty} \phi(x, z)\, e^{i\alpha x}\, dx, \quad \alpha = \sigma + i\tau$$

$$= \int_{-\infty}^{0} \phi(x, z)\, e^{i\alpha x}\, dx + \int_{0}^{\infty} \phi(x, z)\, e^{i\alpha x}\, dx$$

$$= \bar{\phi}_-(\alpha, z) + \bar{\phi}_+(\alpha, z). \qquad \text{... (18)}$$

We assume that for given z, $|\phi|$, $|\phi_1|$ and $|\Psi_1|$ are bounded by $M \exp(-d\,|x|)$ as $|x| \to \infty$, $M, d > 0$, then $\bar{\phi}_+(\alpha, z)$, $\bar{\phi}_-(\alpha, z)$ and $\bar{\phi}(\alpha, z)$ (along with their derivatives) are analytic in $\tau > -d$, $\tau < d$ and $-d < \tau < d$ respectively of the complex α-plane. The Fourier transforms $\bar{\phi}_1(\alpha, z)$, $\bar{\Psi}_1(\alpha, z)$ of $\phi_1(x, z)$ and $\Psi_1(x, z)$ have a similar behaviour.

4. Solution of the Problem

We take Fourier transforms of eqns. (3)-(5) and obtain their solutions to be

$$\bar{\phi}(\alpha, z) = A(\alpha)e^{-\gamma z} + B(\alpha)\, e^{\gamma z}, \quad \gamma^2 = \alpha^2 - k^2 \qquad \text{... (19)}$$

$$\bar{\phi}_1(\alpha, z) = C(\alpha)e^{-\gamma_1 z}, \quad \gamma_1^2 = \alpha^2 - k'^2 \qquad \text{... (20)}$$

$$\bar{\Psi}_1(\alpha, z) = D(\alpha)\, e^{-\delta_1 z}, \quad \delta_1^2 = \alpha^2 - k''^2. \qquad \text{... (21)}$$

Real parts of γ, γ_1 and δ_1 are positive. Multiplying (14) and (17) by exp $(i\alpha x)$, integrating (14) from $x = 0$ to $x = \infty$ and (17) from $x = -\infty$ to $x = 0$, we get

$$\bar{\phi}_+(\alpha, -H) = 0 \qquad \text{... (22)}$$

and

$$\bar{\phi}_-(\alpha, -H) + a\,\bar{\phi}'_-(\alpha, -H) = \frac{ai\gamma_N D}{\alpha - \alpha_N}, \quad \alpha \neq \alpha_N. \qquad \text{... (23)}$$

Adding (22) and (23), we get

$$\bar{\phi}(\alpha, -H) = -a\,\bar{\phi}'_-(\alpha, -H) + \frac{ai\gamma_N D}{\alpha - \alpha_N}. \qquad \text{... (24)}$$

Let us now take Fourier transformation of the conditions (16), which give

$$\bar{\phi}'(\alpha, 0) = \bar{\phi}'_1(\alpha, 0) + i\alpha\,\bar{\Psi}_1(\alpha, 0), \qquad \text{... (25)}$$

$$\lambda k^2\,\bar{\phi}(\alpha, 0) = \lambda_1 (k')^2\,\bar{\phi}_1(\alpha, 0) - 2\mu_1 \frac{d^2}{dz^2}(\bar{\Psi}_1(\alpha, 0)) + 2i\alpha\mu\,\bar{\Psi}'_1(\alpha, 0) \qquad \text{... (26)}$$

and

$$2i\alpha\,\bar{\phi}'_1(\alpha, 0) + \alpha^2\,\bar{\Psi}_1(\alpha, 0) + \frac{d^2}{dz^2}(\bar{\Psi}_1(\alpha, 0)) = 0. \qquad \text{... (27)}$$

Using (20)-(21) in (26)-(27), we get

$$2i\alpha\,[\bar{\phi}'_1(\alpha, 0) + i\alpha\,\bar{\Psi}_1(\alpha, 0)] + (k'')^2\,\bar{\Psi}_1(\alpha, 0) = 0, \qquad \text{... (28)}$$

$$\lambda k^2\,\bar{\phi}(\alpha, 0) = (\lambda_1 (k')^2 - 2\mu_1\gamma_1^2)\,\bar{\phi}_1(\alpha, 0) + 2i\alpha\,\delta_1\,\mu_1\,\bar{\Psi}_1(\alpha, 0), \qquad \text{... (29)}$$

λ; λ_1, μ_1 are the elastic constants of the layer and the solid half space respectively. From (25) and (28)-(29), it is obtained that

$$\bar{\Psi}_1(\alpha, 0) = \frac{-2i\alpha}{(k'')^2}\,\bar{\phi}'(\alpha, 0) \qquad \text{... (30)}$$

and

$$\bar{\phi}_1(\alpha, 0) = \frac{1}{(k'')^2\gamma_1}\,\bar{\phi}'(\alpha, 0)\,[2\alpha^2 - (k'')^2]. \qquad \text{... (31)}$$

Using (30)-(31) in (25), we get

$$\lambda k^2\,\bar{\phi}(\alpha, 0) = \frac{\bar{\phi}'(\alpha, 0)\,p(\alpha)}{(k'')^2\gamma_1} \qquad \text{... (32)}$$

where

$$p(\alpha) = -\mu_1\,[2\alpha^2 - (k'')^2 - 4\alpha^2\gamma_1\delta_1]. \qquad \text{... (33)}$$

From (19) and (32), we obtain

$$\frac{B(\alpha)}{A(\alpha)} = \frac{\gamma p(\alpha) + \lambda k^2 (k'')^2\gamma_1}{\gamma p(\alpha) - \lambda k^2 (k'')^2\gamma_1}. \qquad \text{... (34)}$$

We use the notations $\bar{\phi}(\alpha)$, $\bar{\phi}'(\alpha)$ for $\bar{\phi}(\alpha, -H)$, $\bar{\phi}'(\alpha, -H)$. We put $z = -H$ in (19) and in its derivative w.r.t. z and use (34) to find

$$\bar{\phi}(\alpha) = \left[\frac{\gamma p(\alpha) \cosh \gamma H - (\lambda k^2)(k'')^2 \gamma_1 \sinh \gamma H}{-\gamma p(\alpha) \sinh \gamma H + (\lambda k^2)(k'')^2 \gamma_1 \cosh \gamma H} \right] \frac{\bar{\phi}'(\alpha)}{\gamma}. \qquad \ldots (35)$$

From (24) and (35), we get the Wiener-Hopf type functional equation

$$-a \bar{\phi}'_-(\alpha) + \frac{ai\gamma_N D}{(\alpha - \alpha_N)} = \left[\frac{\gamma p(\alpha) \cosh \gamma H - \lambda k^2 (k'')^2 \gamma_1 \sinh \gamma H}{-\gamma p(\alpha) \sinh \gamma H + \lambda k^2 (k'')^2 \gamma_1 \cosh \gamma H} \right] \frac{\bar{\phi}'(\alpha)}{\gamma}. \qquad \ldots (36)$$

We write

$$G(\alpha) = \frac{\gamma p(\alpha) \cosh \gamma H - (\lambda k^2)(k'')^2 \gamma_1 \sinh \gamma H}{-\gamma p(\alpha) \sinh \gamma H + (\lambda k^2)(k'')^2 \gamma_1 \cosh \gamma H}. \qquad \ldots (37)$$

Then (36) gives

$$\gamma \left[-a \bar{\phi}'_-(\alpha) + \frac{ai\gamma_N D}{\alpha - \alpha_N} \right] = G(\alpha) \bar{\phi}'(\alpha) \qquad \ldots (38)$$

or

$$\bar{\phi}'_-(\alpha) K(\alpha) + \bar{\phi}'_+(\alpha) = \frac{ai\gamma\gamma_N D}{G(\alpha)(\alpha - \alpha_N)} \qquad \ldots (39)$$

where

$$K(\alpha) = \left[\frac{a\gamma}{G(\alpha)} + 1 \right]. \qquad \ldots (40)$$

We solve (39) for $\bar{\phi}'_+(\alpha)$ and $\bar{\phi}'_-(\alpha)$ by invoking the Wiener-Hopf technique.

Let

$$G(\alpha) = \frac{M(\alpha)}{N(\alpha)} \qquad \ldots (41)$$

and let $\alpha = \pm \alpha_{1N}$ and $\alpha = \pm \alpha_N$ be the zeros of $M(\alpha)$ and $N(\alpha)$ respectively (Sato[7]). Then

$$G(\alpha) = \prod_{N=1}^{\infty} \left[\frac{\alpha^2 - \alpha_{1N}^2}{\alpha^2 - \alpha_N^2} \right] \frac{L_1(\alpha)}{L_2(\alpha)}, \qquad \ldots (42)$$

where

$$L_1(\alpha) = M(\alpha) / \prod_{N=1}^{\infty} (\alpha^2 - \alpha_{1N}^2), \qquad \ldots (43)$$

and

$$L_2(\alpha) = N(\alpha)/\prod_{N=1}^{\infty} (\alpha^2 - \alpha_N^2), \quad \ldots (44)$$

are non-zero functions of α. If we write

$$L(\alpha) = L_1(\alpha)/L_2(\alpha) = L_+(\alpha)\, L_-(\alpha) \quad \ldots (45)$$

then

$$G(\alpha) = \prod_{N=1}^{\infty} \left[\frac{\alpha^2 - \alpha_{1N}^2}{\alpha^2 - \alpha_N^2} \right] L_+(\alpha)\, L_-(\alpha) = G_+(\alpha)\, G_-(\alpha), \quad \ldots (46)$$

where

$$G_\pm(\alpha) = \prod_{N=1}^{\infty} \frac{(\alpha \pm \alpha_{1N})}{(\alpha \pm \alpha_N)} L_\pm(\alpha). \quad \ldots (47)$$

$G_\pm(\alpha)$ tend to 1 as $|\alpha| \to \infty$. Therefore,

$$K(\alpha) = K_+(\alpha)\, K_-(\alpha) \quad \ldots (48)$$

and $\quad |K_+(\alpha)|,\ |K_-(\alpha)| \sim |\alpha|^{1/2}$, as $|\alpha| \to \infty$. $\quad \ldots (49)$

(39) can now be decomposed as

$$\bar{\phi}'_-(\alpha)\, K_+(\alpha)\, K_-(\alpha) + \bar{\phi}'_+(\alpha) = \frac{ai\gamma\gamma_N D}{G(\alpha)\,(\alpha - \alpha_N)} = \frac{iK(\alpha)\gamma_N D}{\alpha - \alpha_N} - \frac{i\gamma_N D}{\alpha - \alpha_N} \quad \ldots (50)$$

i.e.,

$$\bar{\phi}'_-(\alpha)\, K_-(\alpha) - \frac{iK_-(\alpha)\,\gamma_N D}{\alpha - \alpha_N} + \frac{i\gamma_N D}{K_+(\alpha_N)\,(\alpha - \alpha_N)}$$

$$= -\frac{\bar{\phi}'_+(\alpha)}{K_+(\alpha)} - \frac{i\gamma_N D}{(\alpha - \alpha_N)} \left[\frac{1}{K_+(\alpha)} - \frac{1}{K_+(\alpha_N)} \right]. \quad \ldots (51)$$

The left-hand member of (51) is analytic in $\tau < d$ and right-hand member in $\tau > -d$. By analytic continuation both sides represent entire functions. By Liouville's theorem, the entire function can be shown to be identically zero. Hence,

$$\bar{\phi}'_+(\alpha) = \frac{i\gamma_N DK_+(\alpha)}{(\alpha - \alpha_N)\, K_+(\alpha_N)} - \frac{i\gamma_N D}{\alpha - \alpha_N} \quad \ldots (52)$$

and

$$\bar{\phi}'_-(\alpha) = \frac{i\gamma_N D}{\alpha - \alpha_N} - \frac{i\gamma_N D}{(\alpha - \alpha_N)\, K_-(\alpha)\, K_+(\alpha_N)}. \quad \ldots (53)$$

Using (53) in (38), we get

$$\bar{\phi}'(\alpha) = \frac{ai\gamma\gamma_N D}{G(\alpha)(\alpha-\alpha_N) K_-(\alpha) K_+(\alpha_N)}. \quad \dots (54)$$

From (35) and (54), we get

$$\bar{\phi}(\alpha) = \frac{ai\gamma_N D}{(\alpha-\alpha_N) K_-(\alpha) K_+(\alpha_N)}. \quad \dots (55)$$

Now, we put $z = -H$ in (19) and use (34) to find $\bar{\phi}(\alpha, z)$ such that

$$\bar{\phi}(\alpha, z) = \left[\frac{\gamma p(\alpha)\cosh\gamma z + \lambda k^2 (k'')^2 \gamma_1 \sinh\gamma z}{\gamma p(\alpha)\cosh\gamma H - \lambda k^2 (k'')^2 \gamma_1 \sinh\gamma H}\right]\bar{\phi}(\alpha) \quad \dots (56)$$

where $\bar{\phi}(\alpha)$ is obtained in (55). The inversion of Fourier transform gives

$$\phi(x, z) = \frac{1}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau} \bar{\phi}(\alpha, z)\, e^{-i\alpha x}\, d\alpha \quad \dots (57)$$

where $\bar{\phi}(\alpha, z)$ is already obtained in (56).

5. Evaluation of Integrals

We evaluate the integral (57) along a closed contour in the complex α-plane with $\alpha = \alpha_N$ as an indentation and $\alpha = \pm k, \pm k', \pm k''$ as branch points (Fig. 2). The permissible sheet in the complex plane is to be selected according to the requirements $\mathrm{Re}(\gamma) \geq 0$, $\mathrm{Re}(\gamma_1) \geq 0$ and $\mathrm{Re}(\delta_1) \geq 0$. The cuts, therefore, will be given by the conditions $\mathrm{Re}(\gamma) = 0 = \mathrm{Re}(\gamma_1) = \mathrm{Re}(\delta_1)$. $\mathrm{Re}(\gamma) = 0$ will imply that only $\mathrm{Im}(\gamma)$ exists and γ^2 will be negative. Therefore

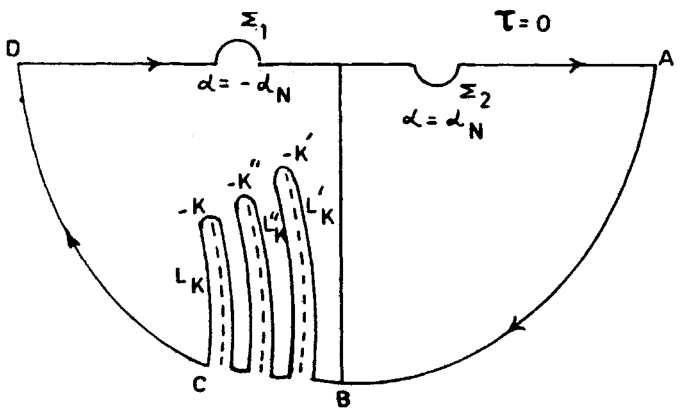

FIG. 2. Contour of integration in the complex α-plane.

$$\gamma^2 = \alpha^2 - k^2 = (\sigma + i\tau)^2 - k^2 < 0$$

or
$$\sigma^2 - \tau^2 + 2i\sigma\tau - (k_1^2 - k_2^2 + 2ik_1 k_2) < 0$$

or
$$\sigma\tau < -k_1 k_2 \text{ and } \sigma^2 - \tau^2 < k_1^2 - k_2^2.$$

The cut is a part of hyperbola in the complex α-plane. The branch cuts at the branch points $\alpha = \pm k, \pm k', \pm k''$ are the parts of hyperbolae as shown in Fig. 2 (Ewing and Press[8]). We have already used the relations that $\mathrm{Re}(\gamma) \geq 0$, $\mathrm{Re}(\gamma_1) \geq 0$ and $\mathrm{Re}(\delta_1) \geq 0$. The factor $\exp(-i\alpha x)$ $(\alpha = \sigma + i\tau)$ makes the integral (57) vanish at infinity in the upper half $(\tau \to \infty)$ of the complex plane if $x < 0$ and in the lower half $(\tau \to -\infty)$ if $x > 0$. If the contour lies in the upper half plane, contribution due to indentation around $\alpha = \alpha_N$ (Roos[9]) is denoted by the potential $\phi_2(x, z)$ (say), then

$$\phi_2(x, z) = -D\, e^{-i\alpha_N x} \sin \gamma_N (z + H). \qquad \dots (58)$$

These are reflected waves in the layer and cancel incident wave. Since $\alpha_N < k$ and so $\gamma^2(\alpha_N) = \alpha_N^2 - k^2 = -\gamma_N^2$. The integrand in (57) can be written as

$$\frac{ai\gamma_N DK_+(\alpha)\,[\gamma p(\alpha) \cosh \gamma z + \lambda k^2 (k'')^2 \gamma_1 \sinh \gamma z]}{(\alpha - \alpha_N) K_+(\alpha_N)\,(a\gamma\{-\gamma p(\alpha) \sinh \gamma H + \lambda k^2 (k'')^2 \gamma_1 \cosh \gamma_H\} + \gamma p(\alpha) \cosh \gamma H - \lambda k^2 (k'')^2 \gamma_1 \sinh \gamma H)}. \qquad \dots (59)$$

Putting the denominator to zero, we obtain

$$\tanh \gamma H = \frac{a\gamma(\lambda k^2)(k'')^2 \gamma_1 - \gamma\mu_1 [(2\alpha^2 - k''^2)^2 - 4\alpha^2 \gamma_1 \delta_1]}{-a\gamma^2 \mu_1 [(2\alpha^2 - k''^2)^2 - 4\alpha^2 \gamma_1 \delta_1] + \lambda k^2 (k'')^2 \gamma_1}. \qquad \dots (60)$$

Equation (60) reduces to (10) when $a = 0$. Schermann[10] has shown that (10) has a finite number of real roots. Equation (60) is the corresponding equation for an impeding surface. Let its real roots be $\alpha = \pm \alpha_s$. Denoting the potential due to the poles at $\alpha = \alpha_s$ by $\phi_3(x, z)$, we find

$$\phi_3(x, z) = \frac{-a\gamma_N D}{K_+(\alpha_N)} \sum_{s=1}^{n} K_+(\alpha_s)\,[e^{i(\gamma_s z - \alpha_s x)} \{\gamma_s p(\alpha_s) - i\lambda k^2 (k'')^2 \gamma_1''\}$$

$$+ e^{-i(\gamma_s z + \alpha_s x)} \{\gamma_s p(\alpha_s) + i\lambda k^2 (k'')^2 \gamma_1''\}]/(\alpha_s - \alpha_N) \left(\frac{dT}{d\alpha}\right)_{\alpha = \alpha_s}, \qquad \dots (61)$$

where

$$\gamma_s = \sqrt{k^2 - \alpha_s^2}, \quad \gamma_1'' = \sqrt{\alpha_s^2 - k'^2} \qquad \dots (62)$$

$$|k'| < |k''| < |\alpha_s| < |k|.$$

These are generalized Rayleigh waves appropriate to the impeding surface. They are confined to the region $x < 0$ and are absent when $a = 0$, where

$$T(\alpha) = a\gamma\,(-\gamma p(\alpha) \sinh \gamma H + \lambda k^2 (k'')^2 \gamma_1 \cosh \gamma H)$$

$$+ \gamma p(\alpha) \cosh \gamma H - \lambda k^2 (k'')^2 \gamma_1 \sinh \gamma H. \qquad \ldots (63)$$

The points $\alpha = \pm k, \pm k', \pm k''$ are branch points. Im(γ), Im(γ_1) and Im(δ_1) change signs on their respective branch cuts. The main contribution is from the close neighbourhood of a branch point. We put $\alpha = k + iu$, u being small and choose the contour in the upper half of the complex plane. We integrate along two sides of the cut and get no contribution. This means that diffracted waves do not propagate with the velocity of the waves of the liquid layer.

Let us now put $\alpha = k' + iu$, Im(γ_1) has opposite signs on the cut as shown in Fig. 2. Further,

$$\gamma_1^2 = (k' + iu)^2 - (k')^2 = 2iu\,(k_1' + ik_2') - u^2$$

i.e.,

$$\gamma_1 = \pm i\sqrt{2k_2' u} = \pm i\gamma_1', \quad k_1' = 0. \qquad \ldots (64)$$

Integrating (57) along two sides of the branch cut at $\alpha = k'$, we get the potential being denoted by $\phi_4(x, z)$

$$\phi_4(x, z) = \frac{ie^{k_2' x}}{2\pi} \int_0^\infty ([\bar{\phi}(\alpha, z)]_{\gamma_1 = i\gamma_1'} - [\bar{\phi}(\alpha, z)]_{\gamma_1 = -i\gamma_1'})\, e^{ux}\, du \qquad \ldots (65)$$

$$= \frac{ie^{k_2' x}}{2\pi} \int_0^\infty \sqrt{u}\, H(u)\, e^{ux}\, du, \quad x < 0, \qquad \ldots (66)$$

where

$$H(u) = \frac{2ai\gamma_N DK_+ (ik_2' + u)\sqrt{2k_2'}\,(\sqrt{(k_2' + u)^2 + k^2})\,\mu_1\,(2(k_2' + u)^2 + k''^2)^2}{(ik_2' + iu - \alpha_N)\, K_+(\alpha_N)\,(C^2 - D^2)}$$

$$\times \Big[\, 8\,(\sqrt{(k_2' + u)^2 + k^2})\,\mu_1 \cos(\sqrt{(k_2' + u)^2 + k^2}\, z(k_2' + u)^2\, (\sqrt{(k_2' + u)^2 + k''^2})$$

$$\times \Big\{ a\,(\sqrt{(k_2' + u)^2 + k^2}) \sin(\sqrt{(k_2' + u)^2 + k^2}\, H + \cos(\sqrt{(k_2' + u)^2 + k^2}\, H \Big\}$$

$$+ \; i(\lambda k^2)(k'')^2\, \{\sin(\sqrt{(k_2' + u)^2 + k^2}\, z\, (a\,(\sqrt{(k_2' + u)^2 + k^2})$$

$$\times \sin(\sqrt{(k_2' + u)^2 + k^2}\, H + \cos(\sqrt{(k_2' + u)^2 + k^2}\, H\,)$$

$$-\; \cos(\sqrt{(k_2' + u)^2 + k^2}\, z\, (\sin(\sqrt{(k_2' + u)^2 + k^2}\, H$$

$$-\, a\,(\sqrt{(k_2' + u)^2 + k^2}) \cos(\sqrt{(k_2' + u)^2 + k^2}\, H\,))\}\Big], \qquad \ldots (67)$$

$$C = -i\left(\sqrt{(k_2'+u)^2+k^2}\right)\mu_1\left(2(k_2'+u)^2+k''^2\right)^2\left[a\sqrt{(k_2'+u)^2+k^2}\right.$$

$$\left.\times\sin\left(\sqrt{(k_2'+u)^2+k^2}\right)H+\cos\left(\sqrt{(k_2'+u)^2+k^2}\right)H\right], \qquad \ldots (68)$$

$$D=\sqrt{2k_2'u}\left[4\mu_1 i\left(\sqrt{(k_2'+u)^2+k^2}\right)(k_2'+u)^2\left(\sqrt{(k_2'+u)^2+k^2}\right)\right.$$

$$\times\left\{a\left(\sqrt{(k_2'+u)^2+k^2}\right)\sin\left(\sqrt{(k_2'+u)^2+k^2}\right)H+\cos\left(\sqrt{(k_2'+u)^2+k^2}\right)H\right\}$$

$$+\lambda k^2(k'')^2\left\{\sin\left(\sqrt{(k_2'+u)^2+k^2}\right)H-a\left(\sqrt{(k_2'+u)^2+k^2}\right)\right.$$

$$\left.\left.\times\cos\left(\sqrt{(k_2'+u)^2+k^2}\right)H\right\}\right]. \qquad \ldots (69)$$

In (66), let us use $x = -y$, where x is negative. Then (66) reduces to

$$\phi_4(x,z)=\frac{-ie^{-k_2'y}}{2\pi}\int_0^\infty\sqrt{u}\,H(u)\,e^{-uy}\,du. \qquad \ldots (70)$$

Integral (70) is evaluated by using the result [Ewing and Press[8]],

$$\int_0^\infty\sqrt{u}\,H(u)\,e^{-ux}\,du=\frac{H(0)\,\Gamma(3/2)}{x^{3/2}}+\frac{H(0)\,\Gamma(5/2)}{x^{5/2}}+\ldots \qquad \ldots (71)$$

$\Gamma(x)$ is the Gamma function of x. Expanding $H(u)$ around $u = 0$ and retaining first term only, we get

$$\phi_4(y,z)\sim\frac{a\,\Gamma(3/2)}{\pi(y)^{3/2}}\,e^{-k_2'y}\,H_1(0)\left[L_1\cos\gamma' z+L_2\sin\gamma' z\right] \qquad \ldots (72)$$

where

$$H_1(0)=-\left(\sqrt{2k_2'}\right)K_+(ik_2')\,\gamma_N\,D/(ik_2'-\alpha_N)\,K_+(\alpha_N)\,\mu_1\,\gamma'$$

$$(2k_2'^2+k''^2)^2\left[a\,\gamma''\sin\gamma''H+\cos\gamma''H\right]^2 \qquad \ldots (73)$$

and

$$L_1=8\gamma''\mu_1(k_2')^2\sqrt{k_2'^2+k''^2}\,(a\,\gamma''\sin\gamma''H+\cos\gamma''H)$$

$$+\;i(\lambda k^2)(k'')^2(\sin\gamma''H-a\,\gamma''\cos\gamma''H) \qquad \ldots (74)$$

$$L_2=i\lambda\,k^2(k'')^2(a\,\gamma''\sin\gamma''H+\cos\gamma''H) \qquad \ldots (75)$$

$$\gamma''=\sqrt{k_2'^2+k^2}. \qquad \ldots (76)$$

Also (72) can be written as

$$\phi_4(y, z) \sim \frac{a\,\Gamma(3/2)}{\pi(y)^{3/2}} e^{-k_2' y} H_1(0)\,[8\gamma''\mu_1 (k_2')^2 \sqrt{(k_2'^2 + k''^2)}$$

$$\times (a\gamma'' \sin\gamma'' H \cos\gamma'' z + \cos\gamma'' H \cos\gamma'' z)$$

$$+\ i\lambda k^2 (k'')^2 (-a\gamma'' \cos\gamma'' (H+z) + \sin\gamma'' (H+z))]. \quad \ldots (77)$$

These are scattered waves.

Now, we put $\alpha = -k - iu$, u being small and choose the contour in lower half. We integrate along two sides of the cut and get no contribution. Let us now put $\alpha = -k' - iu$, Im (γ_1) has opposite signs on the cut as shown in Fig. 2. Further,

$$\gamma_1^2 = (k' + iu)^2 - (k')^2$$

$$= 2iu\,(k_1' + ik_2') - u^2$$

i.e.,
$$\gamma_1 = \pm i\sqrt{2k_2' u} = \pm i\gamma_1', \quad k_1' = 0. \quad \ldots (78)$$

Then, the corresponding potential is

$$\phi_5(x, z) = \frac{ie^{-k_2' x}}{2\pi} \int_0^\infty ([\bar{\phi}(\alpha, z)]_{\gamma_1 = i\gamma_1} - [\bar{\phi}(\alpha, z)]_{\gamma_1 = -i\gamma_1'})\, e^{-ux}\, du \quad \ldots (79)$$

$$= \frac{ie^{-k_2' x}}{2\pi} \int_0^\infty \sqrt{u}\, Q(u)\, e^{-ux}\, du, \quad x > 0, \quad \ldots (80)$$

where

$$Q(u) = 2a\gamma_N D(\sqrt{2k_2'})\,\mu_1 \lambda k^2 (k'')^2 \sqrt{(k_2' + u)^2 + k^2}\ (2(k_2' + u)^2 + k''^2)^2$$

$$\times\ [\sin\sqrt{(k_2' + u)^2 + k^2}\,(z + H)]/(E^2 - F^2)\,(i(k_2' + u) + \alpha_N)$$

$$\times\ K_-(-(ik_2' + u))\, K_+(\alpha_N), \quad \ldots (81)$$

$$E = -i\mu_1 (\sqrt{(k_2' + u)^2 + k^2})\,(2(k_2' + u)^2 + k''^2)^2 \cos(\sqrt{(k_2' + u)^2 + k^2})\, H, \quad \ldots (82)$$

$$F = \sqrt{2k_2' u}\,[4i\mu_1 \sqrt{(k_2' + u)^2 + k^2}\,(k_2' + u)^2 \sqrt{(k_2' + u)^2 + k''^2}$$

$$\times \cos(\sqrt{(k_2' + u)^2 + k^2})\, H + \lambda k^2 (k'')^2 \sin(\sqrt{(k_2' + u)^2 + k^2})\, H]. \ldots (83)$$

The integral (80) is evaluated by using the result in (71) to find

$$\phi_5(x, z) \sim \frac{a\,\Gamma(3/2)\, e^{-k_2' x}}{\pi (x)^{3/2}}\, Q(0) \sin \gamma''(z+H), \quad \dots (84)$$

where

$$Q(0) = \frac{\gamma_N D \sqrt{(2k_2')}\,\lambda\, k^2 (k'')^2}{\mu_1\, i\gamma''\,(2k_2'^2 + k''^2)^2 \cos\gamma''\, H\,(ik_2' + \alpha_N)\, K_-(-ik_2')\, K_+(\alpha_N)}. \quad \dots (85)$$

If $a = 0$, i.e., when there is no pack-ice in the surface, no scattered waves are obtained. The total potential $\phi_t(x, z)$ in the layer is obtained by summing the potentials $\phi_1; \phi_2, \phi_3, \phi_4, \phi_5$ and the contributions of the potentials at the poles $-\alpha_N, -\alpha_S$ and the branch points $\pm k''$. These contribute the incident, reflected, refracted and scattered waves.

6. Conclusions

The scattered waves are of the form $\sin \gamma''(z + H) \exp(-k_2' x)/x^{3/2}$. These are surface waves confined to the layer. They die out exponentially as they move away from the edge of the ice. Similar waves are obtained when contribution from the branch line integral around $\alpha = \pm k''$ is considered. The waves transmitted to the solid half space can be found by evaluating $\bar{\phi}_1(\alpha, z)$ and $\bar{\Psi}_1(\alpha, z)$. The surface waves travelling along the impeding surface are given by (61). The reflected waves are

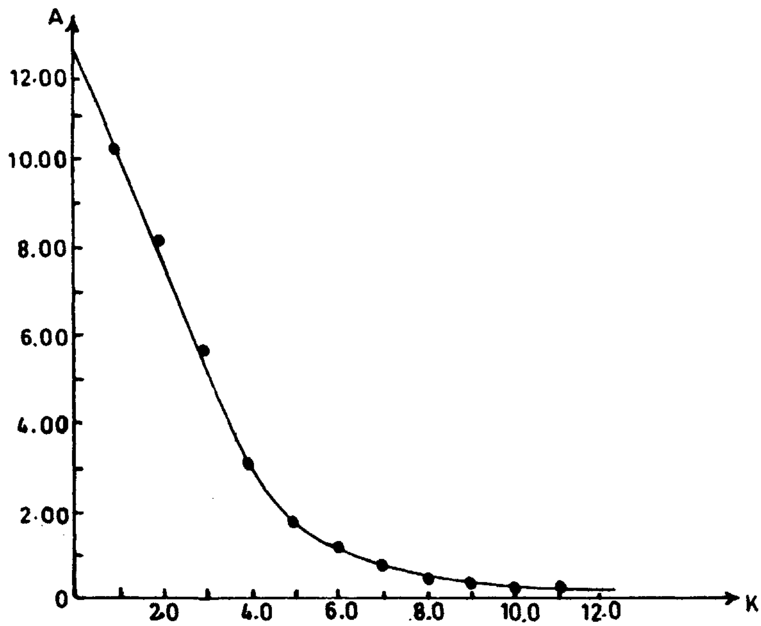

FIG. 3. Amplitude of the scattered wave versus the wave number K.

obtained in (58). The scattered waves propagate with the velocities of the waves in the solid and decay as they move away from the tip of the impeding surface.

When $a = 0$, then $\phi(x, z) = 0$, i.e., when there is no pack-ice on the surface, no scattered, no transmitted and no waves along the impeding surface are obtained. We are left with incident and the waves reflected from the free surface. Numerical computations for the amplitude of the scattered wave (84) have been obtained in terms of the wave number for Poisson's solid for which $\alpha = \sqrt{3}\beta$, for $z = -H$, $H = 5$ km, $a = 1$ km. The amplitude (Fig. 3) falls off rapidly as the wave number increases slowly. As the wave number increases, the wave length of the waves decreases. As the wave-length goes on decreasing, the amplitide of the scattered wave starts falling rapidly.

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