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REPRESENTATION OF FINITE SEMILATTICES BY MEET-IRREDUCIBLES

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It is proved that every finite meet or join-semilattice can be obtained as a collection of levels of a particular fuzzy set, constructed on a poset of its meet-irreducible elements.

1. PRELIMINARIES

A meet-semilattice is here considered to be a partially ordered set in which the infimum exists for every subset with two elements. Similarly, a poset in which there is the supremum for each pair of elements, is said to be a join-semilattice. In other words, a meet-semilattice on a poset $(S, \leq)$ is a commutative, idempotent semigroup (band) $(S, \wedge)$, where the operation $\wedge$ (meet) is the infimum under $\leq$. Analogously, a join-semilattice is an algebra $(S, \vee)$ (commutative band) defined on a poset $(S, \leq)$, in which the operation $\vee$ (join) is the supremum under the order in a poset. Note that we do not consider a semilattice without a partial order which determines the operation.

All semilattices that we use in the paper, are supposed to be finite.

A finite meet-semilattice contains a bottom element (smallest, first, denoted by 0), and in a finite join-semilattice there is a top element (greatest, last, denoted by 1). Obviously, if a finite meet-semilattice contains a top element, then it is a lattice; similarly, a finite join-semilattice with a bottom element is also a lattice.

Finite meet (join)-semilattices (as posets) can be represented by Hasse-diagrams.

Let G be an arbitrary nonempty set, and $(S, \wedge)$ a meet-semilattice. Any function $\overline{A} : G \to S$ is a meet-fuzzy set on G. If $(S, \vee)$ is a join-semilattice, then $\overline{A} : G \to S$ is said to be a join-fuzzy set on G. Further on, for $p \in S$, let $\overline{A}_p : G \to \{0, 1\}$, so that for $x \in G$, $\overline{A}_p(x) = 1$ if and only if $\overline{A}(x) \geq p$, where $\overline{A}$ is a meet (join)-fuzzy set on G. $\overline{A}_p$ is a characteristic function of a p-level subset (or a p-cut) $A_p = \{x \mid \overline{A}_p(x) = 1\}$, of $\overline{A}$.

Semilattices are posets and hence semilattice valued fuzzy sets satisfy some known properties, the proofs of which can be found in Šešelja[7, 11].

Lemma 1 — If $\overline{A} : G \rightarrow S$ is a meet (join)-fuzzy set on G, then for every $x \in G$

$$\overline{A}(x) = \bigvee (p \in S \mid \overline{A}_p\ (x) = 1)$$

(i.e. the supremum on the right exists and is equal to $\overline{A}(x)$.)

Lemma 2 — For a meet-fuzzy set $\overline{A} : G \rightarrow S$

(1) $p \leq q$ implies $A_q \subseteq A_p$;

(2) If $S_1 \subseteq S$ and the supremum of S_1 exists, then

$$\bigcap (A_p \mid p \in S_1) = A_{\vee(p \mid p \in S_1)} \ ;$$

(3) $\bigcup (A_p \mid p \in S) = G$;

(4) for every $x \in G$, $\bigcap (A_p \mid x \in A_p) \in \overline{A}_S$,

where $\overline{A}_S = \{A_p \mid p \in S\}$.

Remark : For join-fuzzy sets (1), (3) and (4) hold as well as for a meet-fuzzy set, and instead of (2) the following is true :

(2′) for $S_1 \subseteq S$

$$\bigcap (A_p \mid p \in S_1) = A_{\vee (p \mid p \in S_1)}.$$

For a meet (join)-fuzzy set $\overline{A} : G \rightarrow S$, define a binary relation $\approx$ on S, such that for $p, q \in S$

$$p \approx q \text{ if and only if } A_p = A_q.$$

$\approx$ is an equivalence relation on S, satisfying the following property.

Lemma 3 — If S is a meet (join)-semilattice and $\overline{A} : G \rightarrow S$ is a meet (join)-fuzzy set on G, then for $p, q \in S$

$$p \approx q \text{ if and only if } [p) \cap \overline{A}(G) = [q) \cap \overline{A}(G),$$

where $[x) = \{y \in S \mid x \leq y\}$, and

$$\overline{A}(G) = \{s \in S \mid s = \overline{A}(x) \text{ for some } x \in G\};$$

Lemma 4 — Let G be a nonempty set, S a collection of its subsets union of which is G, and such that $(S, \subseteq)$ is a join-semilattice. Further on, let for every $x \in G$

$$\bigcap (p \in S \mid x \in p) \in S. \qquad \text{... (1)}$$

If $\overline{A} : G \to S$ is defined with

$$\overline{A}(x) = \bigcap (p \in S \mid x \in p), \qquad \dots (2)$$

then $\overline{A}$ is a meet fuzzy set, where $(S, \leq)$ is a meet-semilattice with $p \leq q$ if and only if $q \subseteq p$ $(p, q \in S)$. Moreover, for every $p \in S$,

$$p = A_p.$$

Lemma 5 — Let G be a nonempty set and S a family of its subsets such that $(S, \subseteq)$ is a meet-semilattice, and $\bigcup S = G$ (meet is the set intersection).

Let $\overline{A} : G \to S$ be defined with

$$\overline{A}(x) = \bigcap (p \in S \mid x \in p).$$

Then, $\overline{A}$ is a join-fuzzy set, where $(S, \leq)$ is a join-semilattice, with $p \leq q$ if and only if $q \subseteq p$ $(p, q \in S)$. Moreover, for every $p \in S$,

$$p = A_p.$$

2. Representations by Meet-Irreducible Elements

Let S be a finite meet (join)-semilattice. An element $a \in S$, distinct from the top element of the semilattice (if it exists), is meet-irreducible if and only if $a = b \wedge c$ implies $a = b$ or $a = c$.

It is known that every element of a finite meet (join)-semilattice can be represented as an infimum of meet-irreducible elements.

The following proposition proves that a fuzzy set induces an isotone (order preserving) mapping from the poset of images into $\mathbf{2} = (\{0, 1\}, \leq)$.

Lemma 6 — Let $\overline{A} : G \to S$ be a meet (join)-fuzzy set. If $p \in S$ and $\overline{A}_p$ is a p-cut of $\overline{A}$ then for all $x, y \in G$

$$\overline{A}(x) \geq \overline{A}(y) \text{ implies } \overline{A}_p(x) \geq \overline{A}_p(y).$$

Proof : Let $\overline{A}(x) \geq \overline{A}(y)$. Now, if $\overline{A}_p(y) = 1$, then $\overline{A}(y) \geq p$, hence $\overline{A}(x) \geq p$ and thus $\overline{A}_p(x) = 1$. Thereby, $\overline{A}_p(x) \geq \overline{A}_p(y)$. □

Corollary 1 — For a meet (join)-fuzzy set $\overline{A} : G \to S$, let

$$\overline{A}(G) = \{q \in S \mid q = \overline{A}(x) \text{ for some } x \in G\}.$$

For $p \in S$ the function $\overline{B}_p : \overline{A}(G) \to \mathbf{2}$, such that for $q \in \overline{A}(G)$

$$\overline{B}_p(q) = 1 \text{ if and only if } q \in \overline{A}(G)$$

is isotone. □

Our aim is to describe a representation of finite semilattices by the decomposition of fuzzy sets. In order to do it for meet-semilattices, we need the following proposition.

Proposition 1 — Let $\overline{A} : G \to S$ be a meet-fuzzy set on G, where S is a finite meet-semilattice. All the p-cuts of $\overline{A}$ are distinct if and only if the following holds: at most one meet-irreducible element of S is not in $\overline{A}(G)$, and such an element, if it exists, is maximal in S; further on, all meet irreducible elements from the semilattice $S \setminus \{m\}$ also belong to $\overline{A}(G)$ (m is maximal).

PROOF : Let all meet irreducible elements from S belong to $\overline{A}(G)$. Every element of S except 1, if it exists, is a meet of some meet irreducible elements from S, and thus every element x from S is a meet of all meet irreducible elements of S which are greater than x. Thereby the set of all meet irreducible elements greater than x differs from the corresponding set of elements greater than $y \in S (y \neq x)$ and hence

$$[x) \cap \overline{A}(G) \neq [y) \cap \overline{A}(G), \quad \text{i.e. } \overline{A}_x \neq \overline{A}_y.$$

If there is a meet irreducible element m not belonging to $\overline{A}(G)$, by conditions of the Proposition it is the only one. Moreover, it is maximal, and all meet irreducibles from $S \setminus \{m\}$ are in $\overline{A}(G)$. Consider the function $\overline{A}_1 : G \to S \setminus \{m\}$, defined by $\overline{A}_1(x) = \overline{A}(x)$, for all $x \in G$. It is a meet-fuzzy set, since $(S \setminus \{m\}, \wedge_1)$ is obviously a meet-semilattice, where $\wedge_1$ is a restriction of the operation $\wedge$. By the first part of this proof, all the p-cuts of $\overline{A}_1$ are distinct. (Since m is not in $\overline{A}(G)$, all the p cuts of $\overline{A}$, for $p \neq m$ are the same as those of $\overline{A}_1$, so they are distinct, too). No level subset of $\overline{A}_1$ is equal to $\emptyset$, since all maximal elements except m belong to $\overline{A}(G)$. However, $\overline{A}_m = \emptyset$, and hence all the p-cuts are distinct.

Conversely, suppose that all the p-cuts of $\overline{A}$ are distinct, and that not all meet-irreducibles are in $\overline{A}(G)$.

Let x be a meet-irreducible element of S which is not maximal and which is not in $\overline{A}(G)$. Since S is finite, and x is meet-irreducible, then there is exactly one y covering x. Thus,

$$[y) \cap \overline{A}(G) = [x) \cap \overline{A}(G).$$

If, on the other hand, there are two maximal meet-irreducible elements x and y which are not in $\overline{A}(G)$, then obviously,

$$\overline{A}_x = \overline{A}_y.$$

Suppose that m is maximal element which is not in $\overline{A}(G)$ such that there is a meet irreducible element n of $S \setminus \{m\}$ which is not in $\overline{A}(G)$. Obviously n must be a maximal element in $S \setminus \{m\}$, and $A_n = \emptyset$. Since $A_m = \emptyset$, then

$$\overline{A}_m = \overline{A}_n. \qquad \square$$

Conditions under which levels of a join-fuzzy set are different are similar to the ones for lattices (see Šešelja[8]). We give them in the sequel.

Proposition 2 — Let $\overline{A} : G \to S$ be a join-fuzzy set on G, where S is a finite meet-semilattice. All the p-cuts of $\overline{A}$ are distinct if and only if all meet irreducible elements from the semilattice S different from 1 belong to $\overline{A}(G)$.

PROOF : Suppose that there is a meet-irreducible element $x \in S, x \neq 1$, which is not in $\bar{A}(G)$. Then, there is a unique $y \in S$, which covers x. Therefore, $[x) \cap \bar{A}(G) = [y) \cap \bar{A}(G)$, and by Lemma 3, $A_x = A_y$.

On the other hand, suppose that all meet-irreducibles of S different from 1 are in $\bar{A}(G)$, and let x and y be two distinct elements from S. If one of them is 1, say x, then there is a meet-irreducible element $z \neq 1$ in a filter $[y)$. Hence, $z \in A_y = [y) \cap \bar{A}(G) \neq [1) \cap \bar{A}(G) = A_x$, since the latter is, by assumption, an empty set. Further on, if both x and y are distinct from 1, then each of them is a meet of meet-irreducibles above it. Now, since $x \neq y$, $[x) \cap \bar{A}(G) \neq [y) \cap \bar{A}(G)$, and $A_x \neq A_y$. □

Now we can prove that every finite meet (join)-semilattice is isomorphic to the semilattice of levels of a particular fuzzy set. Namely, the following proposition is a representation theorem for a finite meet (join)-semilattice, by means of isotone functions on the poset of its meet-irreducible elements. These isotone functions are precisely the levels of the fuzzy set, constructed in the proof.

Theorem 1 — Let S be a finite meet (join)-semilattice, and let X be the partially ordered set of its meet-irreducible elements. Let $\bar{A} : X \to S$ be such that for $x \in X$ $\bar{A}(x) = x$. Then

(a) all the p-cuts of $\bar{A}$ are isotone functions from X to **2**; and

(b) The poset of levels $(\bar{A}_S, \subseteq)$ is a join (meet)-semilattice, dually isomorphic with S under $p \mapsto \bar{A}_p$.

PROOF : (a) By Corollary 1.

(b) By Proposition 1 (Proposition 2 for join-semilattices), $f : p \to \bar{A}_p$ is a bijection from S to $\bar{A}_S$. f preserves the dual order, since $p \leq q$ implies $\bar{A}_q \leq \bar{A}_p$. On the other hand, if $\bar{A}_q \leq \bar{A}_p$ i.e. $A_q \subseteq A_p$, then $p \leq q$. Indeed, every element of S is a meet of some meet-irreducible elements. Since $A_q \subseteq A_p$, the corresponding sets of meet-irreducible elements for p and q are in the same relation, hence p and q being their infima (respectively) satisfy $p \leq q$.

Thus, f and f^{-1} preserve the dual order and f is a dual isomorphism. □

Under particular conditions, the construction described in Theorem 1 for meet-semilattices induces a lattice on the poset of levels.

Theorem 2 — Let $(S, \wedge)$ be a finite meet-semilattice, and let X be the partially ordered set of its meet-irreducible elements. Further on, let $x \in X$ be a maximal element in S, covering another meet-irreducible element, and take $X' = X \setminus \{x\}$. Let $\bar{A} : X' \to S$ be such that for $p \in X'$, $\bar{A}(p) = p$.

Then, the poset of levels $(\bar{A}_S, \subseteq)$ is a lattice, in which the semilattice $(S, \wedge)$ is dually embeddable under the mapping $p \to \bar{A}_p$.

PROOF : The only difference with the proof of the preceding theorem is that the level corresponding to the maximal element not included in $\bar{A}(X')$ is empty. This empty set is obviously the bottom element in the join-semilattice of levels, which thus becomes a lattice.

Omitting the order of the bottom element with other minimal elements of the lattice, a join-semilattice is obtained. By the preceding theorem, this semilattice is dually isomorphic with $(S, \wedge)$. □

The above consideration is illustrated by the following example.

Example 1 — Let S be a meet-semilattice given by its Hasse-diagram in Fig. 1.

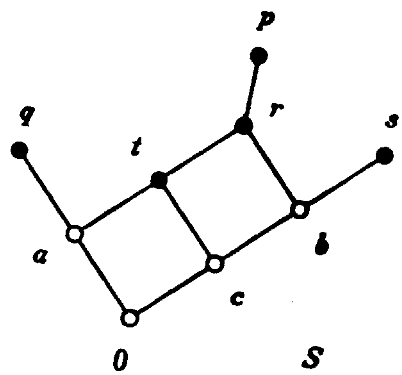

FIG. 1

The set of meet-irreducible elements of S is $X = \{p, q, r, s, t\}$ (these elements are represented by filled circles in the diagram). The meet-fuzzy set $\overline{A} : X \to S$, defined by an identity mapping

$$\overline{A} = \begin{pmatrix} p & q & r & s & t \\ p & q & r & s & t \end{pmatrix}$$

has the following collection of level sets, given by the table of the corresponding characteristic functions :

	p	q	r	s	t
$\overline{A}_p$	1	0	0	0	0
$\overline{A}_q$	0	1	0	0	0
$\overline{A}_r$	1	0	1	0	0
$\overline{A}_s$	0	0	0	1	0
$\overline{A}_t$	1	0	1	0	1
$\overline{A}_a$	1	1	1	0	1
$\overline{A}_b$	1	0	1	1	0
$\overline{A}_c$	1	0	1	1	1
$\overline{A}_0$	1	1	1	1	1

Note that, by Corollary 1, the above level functions are isotone mappings from the semilattice S to the two-element chain $(\{0, 1\}, \leq)$. As in Lemma 4, these levels form a join semilattice (under the set inclusion), which is dually isomorphic to S.

If $X' = X \setminus \{p\}$, then, by Lemma 5, the level functions of a meet-fuzzy set $\overline{A}' : X' \to S$, given again by an identity mapping

$$\overline{A}' = \begin{pmatrix} q & r & s & t \\ q & r & s & t \end{pmatrix}$$

are the following isotone functions from X' to $(\{0, 1\}, \leq)$:

	q	r	s	t
$\overline{A}'_p$	0	0	0	0
$\overline{A}'_q$	1	0	0	0
$\overline{A}'_r$	0	1	0	0
$\overline{A}'_s$	0	0	1	0
$\overline{A}'_t$	0	1	0	1
$\overline{A}'_a$	1	1	0	1
$\overline{A}'_b$	0	1	1	0
$\overline{A}'_c$	0	1	1	1
$\overline{A}'_0$	1	1	1	1

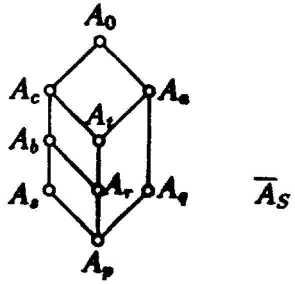

FIG. 2

These isotone functions (or, equivalently, the corresponding level sets) are ordered as the lattice $\overline{A}_S$ by the set inclusion (Fig. 2). Clearly, the join-semilattice S (Fig. 1) is dually embeddable into $\overline{A}_S$, under the mapping $x \mapsto A_x$.

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