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A SHARP CRITERION FOR STARLIKENESS

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The following criterion for starlikeness is obtained. If f is an analytic function in the unit disc U, with $f(0) = 0$ and

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < M_0 = 2.84116..., \text{ for } z \in U,$$

where M_0 satisfies eqn. (8), then f is starlike in U and this result is sharp.

1. INTRODUCTION

In Miller and Mocanu[1] the following sufficient condition for starlikeness was obtained.

If f is an analytic function in the unit disc U, with $f(0) = 0$ and

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < 2, \text{ for } z \in U,$$

then f is starlike. Moreover f satisfies $| zf'(z)/f(z) - 1 | < 1$ in U.

A natural question is to find the maximum value of M for which the inequality

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < M, \text{ for } z \in U \qquad ... (1)$$

implies the starlikeness of f.

In a recent paper the authors[3] proved that in (1) we can take $M = 2.3776$.

In this paper, using a special result on differential subordinations, we give the definitive answer to the above problem, by finding the biggest $M = M_0 = 2.84116...$ such that (1) implies the starlikeness of f. In particular in (1) we can take $M = 2\sqrt{2} = 2.82842...$.

2. Preliminaries

An analytic function f in U, with $f(0) = 0$, is said to be starlike if it is univalent and $f(U)$ is a starlike domain with respect to the origin. It is well-known that f is starlike if and only if $f(0) = 0$, $f'(0) \neq 0$ and

$$\text{Re}\,\frac{zf'(z)}{f(z)} > 0, \quad \text{for } z \in U.$$

If f and g are analytic functions in U and g is univalent, then f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We will need the following subordination result, which is a particular case of a more general result concerning the Briot-Bouquet differential subordinations and can be easily obtained by combining Corollary 1.1 and Theorem 5 in Miller and Mocanu[2].

Lemma — Let h be analytic in U, with $h(0) = 1$ and let q, with $q(0) = 1$, satisfy the differential equation

$$q(z) + \frac{zq'(z)}{q(z)} = h(z). \qquad \text{... (2)}$$

If we suppose that

(a) h is convex in U,

(b) $\text{Re}\, q(z) > 0$, for $z \in U$,

then q is univalent and is given by

$$q(z) = \frac{z\,k'(z)}{k(z)}\,, \qquad \text{... (3)}$$

where

$$k(z) = \int_0^z \frac{g(t)}{t}\,dt \qquad \text{... (4)}$$

and

$$g(z) = z \exp \int_0^z \frac{h(t)-1}{t}\,dt. \qquad \text{... (5)}$$

Moreover if f is analytic in U, with $f(0) = 0$ and

$$1 + \frac{zf''(z)}{f'(z)} \prec h(z) \qquad \text{... (6)}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z). \qquad \text{... (7)}$$

This result is sharp and the extremal function is $f = k$.

3. Main Result

Theorem — If f is an analytic function in U, with $f(0) = 0$, and .

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < M_0 = 2.84116\ldots, \text{ for } z \in U,$$

where M_0 is the unique root in the interval $(2\sqrt{2}, 2.9)$ of equation

$$\sin\left[(M^2+1)\arctan\frac{1}{M} \right] - \left(M - \frac{1}{M} \right)^{M^2} . (M^2+1)^{-(M^2+1)/2} = 0, \qquad \ldots (8)$$

then f is starlike in U. The bound M_0 cannot be replaced by any larger number. The extremal function $f = k$ is given by

$$k(z) = \frac{1}{M_0}\left[\left(1 + \frac{z}{M_0} \right)^{M_0^2} - 1 \right].$$

Proof : The inequality (1) can be rewritten as

$$1 + \frac{zf''(z)}{f'(z)} \prec M\frac{Mz+1}{z+M}, \quad z \in U.$$

If we let $h(z) = M(Mz + 1)/(z + M)$, then from (5) and (4) we obtain

$$g(z) = z\left(1 + \frac{z}{M} \right)^{M^2-1}$$

and

$$k(z) = \frac{1}{M}\left[\left(1 + \frac{z}{M} \right)^{M^2} - 1 \right]$$

respectively.

Hence, according to (3), the solution q of the differential equation (2) is given by

$$q(z) = q_M(z) = \frac{Mz(1+z/M)^{M^2-1}}{(1+z/M)^{M^2}-1}, \quad z \in U. \qquad \ldots (9)$$

Since h is convex, in order to apply the above lemma we only need to put the condition $\operatorname{Re} q(z) > 0$ in U. In this case from (7) we deduce that f is starlike. Therefore our problem is to find the maximum value of M such that

$$\operatorname{Re} q_M(z) \geq 0, \quad \text{for } |z| \leq 1, \qquad \ldots (10)$$

where $q_M(z)$ is given by (9).

We first prove that (10) holds for $M = 2\sqrt{2} = 2.82842\ldots$, which will be a good lower approximation of M_0. If we let

$$\frac{z}{2\sqrt{2}} = \frac{1}{w} - 1$$

and $w = u + iv$, then $|z| = 1$ is equivalent to

$$\frac{u^2 - 2u + v^2 + 1}{u^2 + v^2} = \frac{1}{8}.$$

Hence (10) after some elementary calculations can be rewritten as

$$P(u) = 21952u^7 - 76832u^6 + 112112u^5 - 90888u^4 + 43644u^3$$
$$- 12994u^2 + 1695u + 1695 \geq 0,$$

for $u \in [(8 - 2\sqrt{2})/7, (8 + 2\sqrt{2})/7] \subset [0.7, 1.6]$. For $u < 1$, by grouping the terms of the polynomial P, it is easy to show that $P(u) > 0$. If $u > 1$, then if we let $u = 1 + s$, then

$$P(u) = P(1 + s) = 21952s^7 + 76832s^6 + 112112s^5 + 85512s^4$$
$$+ 32892s^3 + 2242s^2 - 3681s + 384$$

and it is easy to show that the sum of the last six terms is positive (its minimum occurs at $s = s_0 \in (0.14, 0.145)$. Hence $P(u) > 0$ for all u in the specified interval, which yields (10), for $M = 2\sqrt{2}$.

We now deduce eqn. (8) which will give the exact bound $M_0 = 2.84116...$. If we let $z = e^{it}, t \in [-\pi, \pi]$, then the inequality (10) is equivalent to

$$\operatorname{Re}\left[e^{-it} + \frac{1}{M} - e^{-it}\left(1 + \frac{e^{it}}{M}\right)^{1-M^2} \right] \geq 0, \quad t \in [-\pi, \pi],$$

i.e. $F(t, M) \geq 0$, where

$$F(t, M) = \cos t + \frac{1}{M} - \left(1 + \frac{2}{M}\cos t + \frac{1}{M^2}\right)^{(1-M^2)/2}$$
$$\cos\left[(M^2 - 1)\arctan\frac{\sin t}{M + \cos t} + t \right], \qquad \dots (11)$$

$$t \in [-\pi, \pi], \quad M \geq 2\sqrt{2}.$$

Hence M_0 will be obtained as the least $M \geq 2\sqrt{2}$ satisfying the system

$$\begin{cases} F(t, M) = 0 \\ \partial/\partial t\, F(t, M) = 0, \end{cases}$$

for $t \in [0, \pi]$. According to (11) this system becomes

$$\begin{cases} \cos t + \dfrac{1}{M} - [\varphi(t, M)]^{(1-M^2)/2} \cdot \cos \psi(t, M) = 0 \\ -\sin t - \left\{ \dfrac{M^2 - 1}{M} \sin t \cos \psi(t, M) - \chi(t, M). \sin \psi(t, M) \right\} \cdot [\varphi(t, M)]^{-(1+M^2)/2} = 0, \end{cases} \qquad \text{... (12)}$$

where

$$\varphi(t, M) = 1 + \frac{2}{M} \cdot \cos t + \frac{1}{M^2} ,$$

$$\psi(t, M) = (M^2 - 1) \cdot \arctan \frac{\sin t}{M + \cos t} + t$$

and

$$\chi(t, M) = \left(M + \frac{1}{M} \right) \cos t + 2.$$

From (12) we deduce

$$\cos \psi(t, M) = \left(\cos t + \frac{1}{M} \right) \cdot [\varphi(t, M)]^{(M^2-1)/2} \qquad \text{... (13)}$$

and

$$\chi(t, M) \cdot [-[\varphi(t, M)]^{(M^2-1)/2} \cdot \sin t + \sin \psi(t, M)] = 0. \qquad \text{... (14)}$$

We shall show that the second factor in (14) cannot be zero for $M \in [2\sqrt{2}, 3]$. Indeed, if we suppose that this factor vanishes then from (13) we easily deduce $\cos t = -1/(2M)$, which shows that $t \in (\pi/2, 3\pi/4)$ and using again (13) we obtain

$$\frac{1}{2M} - \cos \left[\pi M^2 - (2M^2 - 1) \cdot \arccos \frac{1}{2M} \right] = 0. \qquad \text{... (15)}$$

Hence for $M \in [2\sqrt{2}, 3]$, we have

$$G(M) = \pi M^2 - (2M^2 - 1) \arccos \frac{1}{2M} > \pi M^2 - (2M^2 - 1) \frac{\pi}{2} = \frac{\pi}{2}$$

and

$$G(M) < \pi M^2 - (2M^2 - 1) \arccos \frac{1}{4\sqrt{2}} = 2M^2 \left(\frac{\pi}{2} - \arccos \frac{1}{4\sqrt{2}} \right)$$

$$+ \arccos \frac{1}{4\sqrt{2}} < 9\pi - 17 \arccos \frac{1}{4\sqrt{2}} < \frac{3\pi}{2}.$$

Hence eqn. (15) has no roots in the interval $[2\sqrt{2}, 3]$.

If $\chi(t, M)$ in (14) vanishes, then

$$\cos t = -\frac{2M}{M^2+1}, \quad \sin t = \frac{M^2-1}{M^2+1}, \qquad \dots (16)$$

for $t \in [0, \pi]$ and $M \in [2\sqrt{2}, 2.9]$. After some elementary calculations, from (12) and (16) we obtain the equation (8).

If we denote by

$$F_1(M) = \sin\left[(M^2+1)\arctan\frac{1}{M}\right]$$

$$F_2(M) = \left(M - \frac{1}{M}\right)^{M^2} \cdot (M^2+1)^{-(M^2+1)/2},$$

then the left-hand side of (8) can be written as

$$Q(M) = F_1(M) - F_2(M)$$

and we shall show that the function Q is decreasing and has a unique root M_0 in the interval $[2\sqrt{2}, 2.9]$. We have

$$F_2'(M) = 2M\left(M - \frac{1}{M}\right)^{M^2} \cdot (M^2+1)^{-(M^2+1)/2} H_2(M),$$

where

$$H_2(M) = \frac{1}{M^2-1} + \ln\frac{M^2-1}{M\sqrt{M^2+1}}.$$

Since

$$H_2'(M) = \frac{M^4-4M^2-1}{M(M^2-1)^2(M^2+1)} > 0$$

for $M \in [2\sqrt{2}, 2.9]$ and $H_2(2\sqrt{2}) > -1/20$, $H_2(2.9) = -0.047\ldots < 0$, we easily deduce that $-1/20 < H_2(M) < 0$ and $F_2'(M) > -1/10$, for $M \in [2\sqrt{2}, 2.9]$. On the other hand we have

$$F_1'(M) = H_1(M)\cos\left[(M^2+1)\arctan\frac{1}{M}\right],$$

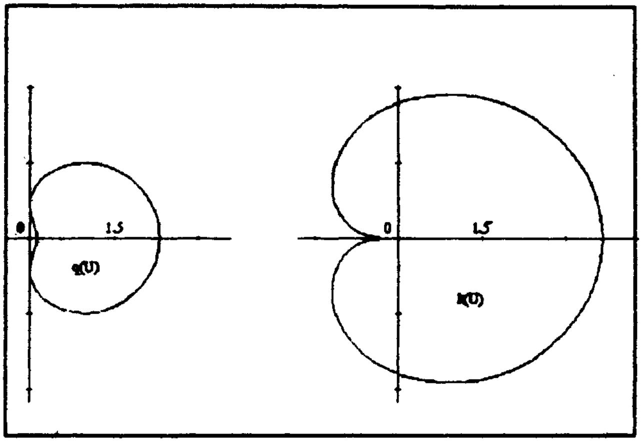
where

$$H_1(M) = 2M\arctan\frac{1}{M} - 1.$$

It is easy to show that H_1 is increasing and $H_1(M) > H_1(2\sqrt{2}) > 0.92$, for $2\sqrt{2} \le M \le 2.9$. We also have $(M^2+1)\arctan 1/M \in (\pi/2, \pi)$ and $\cos[(M^2+1)\arctan 1/M] < \cos\left[9\arctan\frac{1}{2\sqrt{2}}\right] < -0.99$. Hence $F_1'(M) < -0.9$ and finally

$$Q'(M) = F_1'(M) - F_2'(M) < -0.8 < 0, \text{ for } M \in [2\sqrt{2}, 2.9].$$

Since $Q(2\sqrt{2}) = 0.0114... > 0$ and $Q(2.9) = -0.054... < 0$ we deduce that eqn. (8) has a unique root lying in the interval $[2\sqrt{2}, 2.9]$. Using Newton method we obtain the following approximation of this root : $M_0 = 2.841161269...$. The images of the unit disc U by the extremal functions q and k, for $M = M_0$ are given in the following figures.



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