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ON SPHERICALLY SYMMETRIC GENERALIZED THERMOELASTIC WAVES IN A TRANSVERSELY ISOTROPIC MEDIUM†

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The distribution of temperature, displacement and stress in an infinite homogeneous transversely isotropic elastic solid having a spherical cavity has been investigated by taking (i) unit step in stress and zero temperature change, and (ii) unit step in temperature and zero stress, at the boundary of the cavity. The Laplace transform on time has been used to obtain the solutions in the context of generalised thermoelasticity formulated by Noda *et al.* that combines both the theories developed by Lord and Shulman as well as Green and Lindsay. Because of the short duration of "second sound" effects, the small time solutions have been derived. The results have been discussed at the wave fronts. The results obtained theoretically have also been verified numerically and are represented graphically.

1. INTRODUCTION

Recently, the generalized theory of thermoelasticity (Lord and Shulman[1]) has been extended to anisotropic solids by Dhaliwal and Sherief[2]. Singh and Sharma[3] studied the propagation of plane harmonic waves in a generalized thermoelastic homogeneous transversely isotropic medium and Sharma and Sidhu[4] investigated the problem of propagation of generalized thermoelastic waves in homogeneous anisotropic media. Sharma[5] studied transient generalized thermoelastic waves in a transversely isotropic medium with a cylindrical hole, and Sharma and Chand[6] discussed the distribution of temperature and stresses in an elastic plate resulting from a suddenly punched hole. While Wadhawan[7] studied the problem of spherically symmetric thermoelastic disturbances in the context of Lord and Shulman[1] theory (Here in after called L-S theory), Chatterjee and Roychaudhuri[8] investigated the same problem in the context of Green and Lindsay[9] theory (Here in after called G-L Theory).

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In the present paper, the distribution of deformation, temperature and stress in an infinite homogeneous transversely isotropic elastic medium due to (i) unit step in stress and zero temperature change, and (ii) unit step in temperature and zero stress, acting on the boundary of the spherical cavity in the medium, have been investigated in the context of the generalized thermoelasticity formulated by Noda *et al.*[10] that combines both the theories developed in Dhaliwal and Sherief[2] and Green and Lindsay[9], by using Laplace transform technique. Because the "second sound" effects are short lived, so small time approximations have been considered. The results obtained theoretically, have also been verified numerically and are represented graphically.

2. Formulation of the Problem

We consider an infinitely extended homogeneous transversely isotropic thermoelastic solid having a spherical cavity of radius a. Let the origin of the spherical co-ordinate system (r, Θ, ϕ) be at the centre of the cavity. The medium is characterised by $r \geq a$ and we analyse the problem of the transient thermoelastic waves due to (i) a unit step in stress and zero temperature change, and (ii) zero stress and a unit step in temperature, acting on the inner boundary of the spherical cavity. We consider the case of spherical symmetry so that the non-zero displacement component $u = u(r, t)$. Then the governing field equations of motion and heat conduction in the absence of body forces and heat sources are[2, 10] :

$$C_{11}\,[u_{,rr} + (2/r)U_{,r} - (2/r^2)u] - \beta_1\,(T + t_1\,\dot{T}) = \rho\,\ddot{u}, \qquad \text{... (2.1)}$$

$$K_1\,[T_{,rr} + (2/r)T_{,r}] - \rho\,C_e\,(\dot{T} + t_0\,\ddot{T}) = \beta_1\,T_0\,[\,\dot{u}_{,r} + (2/r)\,\dot{u} + t_0\,\delta_{1K}\,(\,\ddot{u}_{,r} + (2/r)\,\ddot{u}\,)] \qquad \text{... (2.2)}$$

where $\beta_1 = (C_{11} + C_{12})\,\alpha_1 + C_{13}\,\alpha_3$, C_{ij} are isothermal elastic parameters, K_1 is the thermal conductivity, ρ, C_e, and t_0, t_1 are respectively the density, specific heat at constant strain and the thermal relaxation times, δ_{1K} is the Kronecker's delta and $\dot{u} = \partial u/\partial t$, $u_{,r} = \partial u/\partial r$ etc.

For L-S theory $t_1 = 0$, $\delta_{1K} = 1$ and for G-L theory $t_1 > 0$, $\delta_{1K} = 0$ (i.e. $K = 1$ for L-S and $K = 2$ for G-L theory). The thermal relaxation times t_1 and t_0 satisfy the inequalities[11]

$$t_1 \geq t_0 \geq 0, \quad \text{for G-L theory only.} \qquad \text{... (2.3)}$$

The medium is assumed to be at rest and undisturbed initially, so the initial and regularity conditions can be written as

$$\left.\begin{aligned} &u = 0 = T,\ \dot{u} = 0 = \dot{T},\ \text{at } t = 0,\ r \geq a,\ \partial u/\partial t = 0,\ \text{at } r = 0 \\ \text{and} \quad &u = 0 = T,\ \text{for } t \geq 0 \text{ when } r \to \infty. \end{aligned}\right\} \qquad \text{... (2.4)}$$

We take two types of boundary conditions for $t > 0$ as :

$$\text{(i)}\ \sigma_{rr}(r, t) = H(t),\quad T(r, t) = 0 \qquad \text{... (2.5)}$$

$$\text{and (ii)}\ \sigma_{rr}(r, t) = 0,\quad T(r, t) = H(t), \quad \text{on } r = a \qquad \text{... (2.6)}$$

where $H(t)$ is Heaviside function of time.

We define the quantities

$$R = w^* r/v, \quad \tau = w^* t, \quad U = \rho w^* vu/\beta_1 T, \quad Z = T/T_0, \quad \tau_0 = w^* t_0,$$

$$\tau_1 = w^* t_1, \quad w^* = C_{11} C_e/K_1, \quad \varepsilon = \beta_1^2 T_0/\rho C_e C_{11}, \quad v^2 = C_{11}/\rho. \quad \text{... (2.7)}$$

Here ε is the thermoelastic coupling constant, w^* the characteristic frequency of the medium and v the velocity of the QL-wave.

Introducing the quantities (2.7) in eqns. (2.1) and (2.2), we obtain

$$U_{,RR} + 2R^{-1} U_{,R} - 2R^{-2} U - \ddot{U} = (Z + \tau_1 \dot{Z})_{,R} \quad \text{... (2.8)}$$

$$Z_{,RR} + 2R^{-1} Z_{,R} - (\dot{Z} + \tau_0 \ddot{Z}) = \varepsilon (\partial/\partial R + 2/R)(\dot{U} + \tau_0 \delta_{1K} \ddot{U}). \quad \text{... (2.9)}$$

The boundary of the spherical cavity i.e. $r = a$ is given by

$$R = w^* a/v = \eta \quad \text{(say)}.$$

The initial and regularity conditions (2.4) become

$$\left.\begin{array}{l} U = 0 = Z \text{ at } \tau = 0,\ R \geq \eta\ ;\ \partial u/\partial \tau = 0 \text{ at } R = 0, \\ \text{and} \quad U = 0 = Z \text{ for } \tau = 0 \text{ when } r \to \infty. \end{array}\right\} \quad \text{... (2.10)}$$

The boundary conditions (2.5) and (2.6) take the form

$$\text{(i)} \quad \sigma_R(\eta, \tau) = H(\tau), \quad Z(\eta, \tau) = 0 \quad \text{... (2.11)}$$

and

$$\text{(ii)} \quad \sigma_R(\eta, \tau) = 0, \quad Z(\eta, \tau) = H(\tau) \quad \text{... (2.12)}$$

where

$$\sigma_R = U_{,R} + 2R^{-1} bU - (Z + \tau_1 \dot{Z}), \quad b = C_{12}/C_{11}. \quad \text{... (2.13)}$$

3. Solution of the Problem

Applying the Laplace transform defined by

$$\bar{\phi}(R, P) = \int_0^\infty \phi(R, \tau) \exp(-p\tau)\, d\tau \quad \text{... (3.1)}$$

w.r.t. time, to eqns. (2.8) and (2.9) and using (2.10), we get

$$\{D(D + 2R^{-1}) - p^2\}\, \bar{U} = p^2 \tau_1^* D \bar{Z} \quad \text{... (3.2)}$$

$$\{D + 2R^{-1})D - \tau_0^* p^2\}\, \bar{Z} = \varepsilon\, \tau_0' p^2 (D + 2R^{-1})\, \bar{U} \quad \text{... (3.3)}$$

where

$$\tau_1^* = \tau_1 + p^{-1}\ ,\ \tau_0^* = \tau_0 + p^{-1}\ ,\ \tau_0' = \tau_0\,\delta_{1K} + p^{-1}\ ,\ D = d/dr. \qquad \text{... (3.4)}$$

Equations (3.2) and (3.3) provide us

$$[\{D(D+2R^{-1})\}^2 - (m_1^2+m_2^2)\,D(D+2R^{-1}) + m_1^2 + m_2^2\,]\,\overline{U} = 0 \qquad \text{... (3.5)}$$

$$\{(D+2R^{-1})\,D\}^2 - (m_1^2+m_2^2)\,(D+2R^{-1})D + m_1^2\,m_2^2\,]\,\overline{Z} = 0 \qquad \text{... (3.6)}$$

where m_1^2 and m_2^2 are given by the quadratic equation

$$m^4 - p[1+\varepsilon + p\,(1+\tau_0+\varepsilon\,\tau_0\,\delta_{1K}+\varepsilon\tau_1)\,]\,m^2 + p^3 + \tau_0\,p^4 = 0. \qquad \text{... (3.7)}$$

The solution of eqns. (3.5) and (3.6) are modified Bessel's functions of order 3/2 and 1/2 respectively (Watson[12], and can be expressed in terms of exponentials so that

$$\overline{U} = \sum_{i=1}^{2} A_i\,(1/R + 1/m_i\,R^2)\exp(-m_i\,R) \qquad \text{... (3.8)}$$

$$\bar{\bar{Z}} = \sum_{i=1}^{2} B_i \exp(-m_i\,R)/R \qquad \text{... (3.9)}$$

where A_i and B_i are constants.

Using eqns. (3.8) and (3.9) in (3.3), we get

$$B_i = -\,((m_i\,\varepsilon p\,(\tau_0\,\delta_{1K}\,p+1)\,A_i/(m_i^2 - (\tau_0\,p+1)p). \qquad \text{... (3.10)}$$

Case I : Step input of stress and constant temperature

We consider a constant stress of magnitude unity suddenly applied on the boundary of the spherical cavity and the temperature at the boundary is kept constant, in this case eqns. (3.8)-(3.10) along with the boundary conditions (2.11) lead to

$$A_1 = Y_2/p\,\Delta,\quad A_2 = -\,Y_1/p\,\Delta \qquad \text{... (3.11)}$$

where $\Delta = X_1\,Y_2 - X_2\,Y_1$ ... (3.12)

$$\left.\begin{aligned} X_i &= \{a(1+m_i\,\eta) + m_i^2\,\eta^2\}\ \exp\,(-m_i\,\eta)/m_i\,\eta \\ Y_i &= -\ \varepsilon p\,(\tau_0\,\delta_{1K}\,p+1)\,m_i \exp(-m_i\eta)/(m_i^2 - (\tau_0\,p+1)p) \end{aligned}\right\} \qquad \text{... (3.13)}$$

$$a = 2(1-b) = 2(C_{11} - C_{12})/C_{11}. \qquad \text{... (3.14)}$$

Case II : Step input of temperature and zero stress

We consider a constant temperature of magnitude unity suddenly applied on the boundary of the spherical cavity and the boundary is kept stress free. In this case eqns. (3.8)-(3.10) alongwith the boundary conditions (2.12) lead to

$$A_1 = [X_2 - (\tau_1 p + 1)\, pY_2]/p\Delta$$

$$A_2 = [X_1 - (\tau_1 p + 1)\, pY_1]/p\Delta \qquad \ldots (3.15)$$

where Δ is given by eqn. (3.12) and X_i, Y_i are given by (3.13).

4. Small Time-Approximations

Because the 'second sound' effects are short lived (Green[11]), therefore we concentrate our attention on small time approximations i.e. we take p large. The roots m_i, i = 1, 2 of eqn. (3.7) can be approximated as[13]

$$m_i = pv_i^{-1} + \phi_i + o(p^{-1}), \quad i = 1, 2 \qquad \ldots (4.1)$$

where

$$v_{1,2}^{-1} = (K_2 \pm \sqrt{M})^{1/2}/\sqrt{2},$$

$$\phi_{1,2} = [K_1 \pm (K_1 K_2 - 2)/\sqrt{M}\,]/2\sqrt{2}\,(K_2 \pm \sqrt{M})^{1/2} \qquad \ldots (4.2)$$

$$M \doteq K_2^2 - 4\tau_0, \quad K_1 = 1 + \varepsilon, \; K_2 = 1 + \tau_0 + \varepsilon\tau_1 + \varepsilon\tau_0\,\delta_{1K}. \qquad \ldots (4.3)$$

Again $\quad M = (1 + \varepsilon\tau_1 - \tau_0)^2 + 4\varepsilon\tau_0\,\tau_1 + \varepsilon\,\tau_0\,\delta_{1K}\,\{\varepsilon\tau_0 + 2(1 + \tau_0 + \varepsilon\tau_1)\} > 0.$

Also $(1 + \varepsilon\tau_1 + \tau_0 + \varepsilon\tau_0\,\delta_{1K})^2 > M$ so that $v_1 < v_2$. Thus v_1 corresponds to the speed of slowest wave and v_2 to that of fastest wave. As a consequence of this, the points of the solid for which $R > \tau\, v_2$ dot not experience any disturbance. From eqns. (4.2) and (4.3) we see that as $\tau_0 = \tau_1 \to 0$; $v_1 \to 1$ and $v_2 \to \infty$. But $\tau_1 = \tau_0 = 0$ corresponds to the case of the coupled theory of thermoelasticity, which predicts an infinite speed of heat propagation. We conclude that the wave propagating with speed v_1 must be elastic influenced by the thermal field. Since $v_1 < v_2$, the elastic wave follows the thermal wave.

For Case I, using expansion (4.1) in (3.13) and (3.12) and then in (3.8) to (3.9), we obtain

$$\bar{Z}(R, p) = (\varepsilon\,\tau_0\, v_1^2\, v_2^2\,)/R(v_1^2 - v_2^2\,)$$

$$\times \{\exp\,(-\,\phi_1 R_1)\,[(1/p) + (M_1' + S_1')\,(1/p^2)]\;\exp\; -pR_1/v_1)$$

$$+ \exp\,(-\,\phi_2 R_1\,[1/p + (M_2' + S_2')\,1/p^2\,]\,\exp\,(-pR_1/v_2\} \qquad \ldots (4.4)$$

$$\overline{U}(R,\ p) = ((-\nu_1 \nu_2 \eta)/R(\nu_1^2-\nu_2^2))\ \{\nu_2 (1-\tau_0 \nu_1^2) \exp(-\phi_1 R_1) \times [(1/p^2)+(\nu_1/R+M_1')/p^3] \exp(-pR_1/\nu_1) + \nu_1 (1-\tau_0 \nu_2^2) \exp(-\phi_2 R_1)\ [1/p^2+(\nu_2/R+M_2')/p^3] \times \exp(-pR_1/\nu_2)\} \quad \text{... (4.5)}$$

$$\bar{\sigma}_R = ((-\nu_1^2 \nu_2^2)/R(\nu_1^2-\nu_2^2))\ \{[(1-\tau_0 \nu_1^2+\varepsilon \tau_0 \nu_1^2)/\nu_1^2 p + M_1''/p^2] \times \exp(-(\phi_1+p/\nu_1) R_1) + [(1-\tau_0 \nu_2^2+\varepsilon \tau_0 \nu_2^2)/\nu_2^2 p + M_2''/p^2] \times \exp(-(\phi_2+p/\nu_2) R_1)\} \quad \text{... (4.6)}$$

for L-S theory and

$$Z(R,\ p) = (-\varepsilon \eta \nu_1^2 \nu_2^2/R(\nu_1^2-\nu_2^2))\ \{\nu_2 (1-\tau_0 \nu_1^2) \exp(-\phi_1 R_1) \times [1/p^2+(M_1+S_1)/p^3] \exp(-p R_1/\nu_1) + \exp(-\phi_2 R_1) \nu_1 (1-\tau_0 \nu_2^2) \times [1/p^2+(M_2+S_2) 1/p^3] \exp(-pR_1/\nu_2)\} \quad \text{... (4.7)}$$

$$\overline{U}(R,\ p) = (\nu_1 \nu_2/R(\nu_1^2-\nu_2^2))\ \{\nu_2 (1-\tau_0 \nu_1^2) \exp(-\phi_1 R_1) \times [(1/p^2)+(\nu_1/R-M_1)/p^3] \exp(-p R_1/\nu_1) + \nu_1 (1-\tau_0 \nu_2^2) \exp(-\phi_2 R_1)\ [(1/p^2)+(\nu_2/R-M_2)/p^3] \times \exp(-pR_1/\nu_2)\} \quad \text{... (4.8)}$$

$$\bar{\sigma}_R = (\nu_1^2 \nu_2^2/R(\nu_1^2-\nu_2^2))\ \{[(1-\tau_0 \nu_1^2+\varepsilon \tau_1 \nu_1^2 \eta)/\nu_1^2 p + M_1^*/p^2] \times \exp(-R_1 (\phi_1+p/\nu_1)) + [(1-\tau_0 \nu_2^2+\varepsilon \tau_1 \nu_2^2 \eta)/\nu_2^2 p + M_2^*/p^2] \times \exp(-R_1 (\phi_2+p/\nu_2))\} \quad \text{... (4.9)}$$

for G-L theory, where

$$S_1 = [\phi_1 \nu_1 (1+\tau_0 \nu_1^2) - \nu_1^2]/(1-\tau_0 \nu_1^2),$$
$$S_2 = [\phi_2 \nu_2 (1+\tau_0 \nu_2^2) - \nu_2^2]/(1-\tau_0 \nu_2^2),$$

$$Q = [N_1 \nu_1^2 - N_2 \nu_2^2 + \tau_0 \nu_1^2 \nu_2^2 (N_1-N_2)]/(\nu_1^2-\nu_2^2),$$

$$N_1 = \phi_2 \nu_2 + a\nu_2/\eta - S_1, \quad N_2 = \phi_1 \nu_1 + a\nu_1/\eta - S_2,$$

$$M_1 = S_2 + Q, \quad M_2 = S_1 + Q, \quad S_1' = \tau_0^{-1} - S_1, \quad S_2' = \tau_0^{-1} - S_2,$$

$$N_1' = N_1 + (\phi_1 v_1 (1-\tau_0) + \tau_0)/\tau_0, \quad N_2' = N_2 + (\phi_2 v_2 (1-\tau_0) + \tau_0)/\tau_0,$$

$$Q' = [N_1' v_1^2 - N_2' v_2^2 + \tau_0 v_1^2 v_2^2 (N_1' - N_2')]/(v_1^2 - v_2^2),$$

$$M_1' = S_2' - Q', \quad M_2' = S_1' - Q', \quad R_1 = R - \eta,$$

$$M_1'' = \{(1 - \tau_0 v_1^2 + \varepsilon\tau_0 v_1^2) M_1' + \varepsilon \tau_0 v_1^2 S_1' + (\phi_1 v_1 - a v_1/R)(1 - \tau_0 v_1^2)\}/v_1^2,$$

$$M_2'' = \{(1 - \tau_0 v_2^2 + \varepsilon\tau_0 v_2^2) M_2' + \varepsilon \tau_0 v_2^2 S_2' + (\phi_2 v_2 - a v_2/R)(1 - \tau_0 v_2^2)\}/v_2^2,$$

$$M_1^* = \{(1 - \tau_0 v_1^2 + \varepsilon \tau_1 \eta v_1^2) M_1 + (1 - \tau_0 v_1^2)(\phi_1 v_1 - a v_1/R) + \varepsilon \eta \tau_1 v_1^2 (S_1 + \tau_1^{-1})\}/v_1^2,$$

$$M_2^* = \{(1 - \tau_0 v_2^2 + \varepsilon \tau_1 \eta v_2^2) M_2 + (1 - \tau_0 v_2^2)(\phi_2 v_2 - a v_2/R) + \varepsilon \eta \tau_1 v_2^2 (S_2 + \tau_2^{-1})\}/v_2^2. \quad \ldots (4.10)$$

Inverting the Laplace transform, we obtain

$$Z(R, \tau) = (\varepsilon \tau_0 v_1^2 v_2^2 / R(v_1^2 - v_2^2)) \{\exp(-\phi_1 R_1)[H(\tau - R_1/v_1) + (M_1' + S_1')(\tau - R_1/v_1) H(\tau - R_1/v_1)] + \exp(-\phi_2 R_1)[H(\tau - R_1/v_2) + (M_2' + S_2') \times (\tau - R_1/v_2) H(\tau - R_1/v_2)]\} \quad \ldots (4.11)$$

$$U(R, \tau) = (v_1 v_2 \eta / R(v_1^2 - v_2^2)) \{v_2 (1 - \tau_0 v_1^2) \exp(-\phi_1 R_1)[(\tau - R_1/v_1) + (v_1/R + M_1')(\tau - R_1/v_1)^2] H(\tau - R_1/v_1) + v_1 (1 - \tau_0 v_2^2) \exp(-\phi_2 R_1)[(\tau - R_1/v_2) + (v_2/R + M_2')(\tau - R_1/v_2)^2] H(\tau - R_1/v_2)\} \quad \ldots (4.12)$$

$$\sigma_R(R, \tau) = (-v_1^2 v_2^2 / R(v_1^2 - v_2^2)) \{\exp(-\phi_1 R_1)[(1 - \tau_0 v_1^2 + \varepsilon \tau_0 v_1^2)/v_1^2 + M_1''(\tau - R_1/v_1)] H(\tau - R_1/v_1) + \exp(-\phi_2 R_1) \times [(1 - \tau_0 v_2^2 + \varepsilon \tau_0 v_2^2)/v_2^2 + M_2''(\tau - R_1/v_2)] H(\tau - R_1/v_2)\} \quad \ldots (4.13)$$

for L-S theory and

$$Z(R, \tau) = (-\varepsilon \eta v_1^2 v_2^2 / R(v_1^2 - v_2^2)) \{\exp(-\phi_1 R_1)[(\tau - R_1/v_1) + (M_1 + S_1) \times (\tau - R_1/v_1)^2] H(\tau - R_1/v_1) + \exp(-\phi_2 R_1)[(\tau - R_1/v_2) + (M_2 + S_2)(\tau - R_1/v_2)^2] H(\tau - R_1/v_2)\} \quad \ldots (4.14)$$

$$U(R,\ \tau) = (v_1 v_2/R(v_1^2 - v_2^2))\ \{v_2 (1 - \tau_0 v_1^2) \exp(-\ \phi_1 R_1) [(\tau - R_1/v_1)$$
$$+\ (v_1/R - M_1)(\tau - R_1/v_1)] H(\tau - R_1/v_1)$$
$$+ v_1 (1 - \tau_0 v_2^2) \exp\ (-\ \phi_2 R_1) [(\tau - R_1/v_2)$$
$$+ (v_2/R - M_2)(\tau - R_1/v_2)] H(\tau - R_1/v_2)\} \quad \dots (4.15)$$

$$\sigma_R (R, \tau) = (v_1^2 v_2^2/R(v_1^2 - v_2^2))\ \{[(1 - \tau_0 v_1^2 + \varepsilon \tau_1 \eta v_1^2)/v_1^2 + M_1^* (\tau - R_1/v_1)]$$
$$\times \exp(-\ \phi_1 R_1) H(\tau - R_1/v_1) + [(1 - \tau_0 v_2^2 + \varepsilon \tau_1 \eta v_2^2)/v_2^2$$
$$+ M_2^* (\tau - R_1/v_2)]\ \exp(-\ \phi_2 R_1)\ H(\tau - R_1/v_2)\} \quad \dots (4.16)$$

for G-L theory.

For Case II, using expansions (4.1) in eqns. (3.13), (3.8)-(3.10) and then in (3.15), we get

$$\bar{Z}\ (R,\ p) = -\ \{v_1^2 (1 - \tau_0 v_2^2 + \varepsilon v_2^2 \eta \tau_0) \exp(-\ \phi_1 R_1)$$
$$\times\ (p^{-1} + \lambda_1' p^{-2}) \exp(-pR_1/v_1) +\ v_2^2 (1 - \tau_0 v_1^2 + \varepsilon v_1^2 \eta \tau_0)$$
$$\times\ \exp(-\ \phi_2 R_1)(p^{-1} + \lambda_2' p^{-2}) \exp(-\ pR_1/v_2)\}/R(v_1^2 - v_2^2)$$
$$\dots (4.17)$$

$$\bar{U}\ (R,\ p) = \{v_1 (1 - \tau_0 v_1^2)(1 - \tau_0 v_2^2 + \varepsilon v_2^2 \eta \tau_0) \exp(-\ \phi_1 R_1)$$
$$\times\ [p^{-2} + (v_1/R + \lambda_1) p^{-3}]\ \exp\ (-pR_1/v_1) + v_2 (1 - \tau_0 v_2^2)$$
$$\times (1 - \tau_0 v_1^2 + \varepsilon v_1^2 \eta \tau_0)\ \exp(-\ \phi_2 R_1) [p^{-2} + (v_2/R + \lambda_2)\ p^{-3}]$$
$$\times\ \exp(-pR_1/v_2)\}\ /\ \varepsilon \tau_0 R (v_1^2 - v_2^2) \quad \dots (4.18)$$

$$\bar{\sigma}_R = (1/R (v_1^2 - v_2^2) \varepsilon \tau_0) \{[(1 - \tau_0 v_2^2 + \varepsilon v_2^2 \eta \tau_0)\ (1 - \tau_0 v_1^2 - \varepsilon \tau_0 v_1^2) p^{-1}$$
$$+\ \lambda_1^* p^{-2}]\ \exp\ (-\ (\phi_1 + p/v_1) R_1) + [(1 - \tau_0 v_1^2 + \varepsilon v_1^2 \eta \tau_0)$$
$$\times\ (1 - \tau_0 v_2^2 - \varepsilon \tau_0 v_2^2)\quad p^{-1} + \lambda_2^* p^{-2}]\ \exp\ (-\ (\phi_2 + p/v_2) R_1\} \dots (4.19)$$

for L-S theory and

$$\bar{Z}\ (R,\ p) = -\ \{v_1^2 (1 - \tau_0 v_2^2 + \varepsilon v_2^2 \eta \tau_1) [p^{-1} + (L_2 - Q - S_1) p^{-2}]$$
$$\times\ \exp(-(\phi_1 + p/v_1) R_1) +\ v_2^2 (1 - \tau_0 v_1^2 + \varepsilon v_1^2 \eta \tau_1)$$
$$\times\ [p^{-1} + (L_1 - Q - S_2) p^{-2}] \exp(-(\phi_2 + p/v_2) R_1)\}/R(v_1^2 - v_2^2)$$
$$\dots (4.20)$$

$$\overline{U}\,(R,\,p) = \{v_1\,(1-\tau_0\,v_1^2)\,(1-\tau_0\,v_2^2+\varepsilon\,\tau_1\,\eta\,v_2^2)$$
$$\times\,[p^{-1}+(v_1/R+L_2-Q)\,p^{-2}]\,\exp\,(-(\phi_1+p/v_1)\,R_1)+v_2\,(1-\tau_0\,v_2^2)$$
$$\times\,(1-\tau_0\,v_1^2+\varepsilon\,\tau_1\,\eta\,v_1^2)\,[p^{-1}+(v_2/R+L_1-Q)\,p^{-2}]$$
$$\times\,\exp\,(-(\phi_2+p/v_2)\,R_1)\}/\varepsilon R\,(v_1^2-v_2^2) \qquad \ldots (4.21)$$

$$\overline{\sigma}_R = \{[(1-\tau_0\,v_2^2+\varepsilon\,v_2^2\,\tau_1\,\eta)\,(1-\tau_0\,v_1^2-\varepsilon\,\tau_1\,v_1^2)+L_1^*/p]$$
$$\times\,\exp\,(-\,(\phi_1+p/v_1)/R_1)\;+\;[1(1\;-\;\tau_0 v_1^2+\varepsilon\,v_1^2\,\tau_1\eta)\,(1-\tau_0 v_2^2-\varepsilon\,\tau_1 v_2^2)$$
$$+\;L_2^*/p]\,\exp\,(-\,(\phi_2+p/v_2)\,R_1)\}/\varepsilon R\,(v_1^2-v_2^2) \qquad \ldots (4.22)$$

for G-L theory, where

$$L_1 = \{(1\;-\;\tau_0\,v_1^2)\,(\phi_1\,v_1-av_1/\eta)$$
$$+\,\varepsilon\,v_1^2\,\eta\,(1-S_1\,\tau_1)\}/(1-\tau_0\,v_1^2+\varepsilon\,\tau_1\,\eta\,v_1^2),$$

$$L_2 = \{(1\;-\;\tau_0\,v_2^2)\,(\phi_2\,v_2-av_2/\eta)$$
$$+\,\varepsilon\,v_2^2\,\eta\,(1-S_2\,\tau_1)\}/(1-\tau_0\,v_2^2+\varepsilon\,\tau_1\,\eta\,v_2^2),$$

$$L_1' = \{(1\;-\;\tau_0\,v_1^2)\,(\phi_1\,v_1-av_1/\eta)+\varepsilon\,v_1^2\,\eta\,\tau_0\,S_1'\}/(1-\tau_0\,v_1^2+\varepsilon\,\eta\,\tau_0\,v_1^2),$$

$$L_2' = \{(1\;-\;\tau_0\,v_2^2)\,(\phi_2\,v_2-av_2/\eta)+\varepsilon\,v_2^2\,\eta\,\tau_0\,S_2'\}/(1-\tau_0\,v_2^2+\varepsilon\,\eta\,\tau_0\,v_2^2),$$

$$\lambda_1 = L_1'-Q',\quad \lambda_2 = L_2'-Q',\quad \lambda_1' = \lambda_1+S_1',\quad \lambda_2' = \lambda_2+S_2',$$

$$\lambda_1^* = (1-\tau_0\,v_2^2+\varepsilon\,\eta\,\tau_0\,v_2^2)\;\{(1-\tau_0\,v_1^2)\,(\phi_1\,v_1-av_1/R+\lambda_1)-\lambda_1'\,(\varepsilon\,\tau_0\,v_1^2)\},$$

$$\lambda_2^* = (1-\tau_0\,v_1^2+\varepsilon\,\eta\,\tau_0\,v_1^2)\;\{(1-\tau_0\,v_2^2)\,(\phi_2\,v_2-av_2/R+\lambda_2)-\lambda_2'\,(\varepsilon\,\tau_0\,v_2^2)\},$$

$$L_1^* = (1-\tau_0\,v_2^2+\varepsilon\,\tau_1\,\eta\,v_2^2)\;\{(1\;-\;\tau_0\,v_1^2-\varepsilon\,\tau_1\,v_1^2)$$
$$(L_2-Q)+(1-\tau_0\,v_1^2)\,(\phi_1\,v_1-av_1/R)\;+\;\varepsilon\tau_1\,v_1^2\,(S_1+\tau_1^{-1})\},$$

$$L_2^* = (1-\tau_0\,v_1^2+\varepsilon\,\tau_1\,\eta\,v^2)\;\{(1\;-\;\tau_0\,v_2^2-\varepsilon\,\tau_1\,v_2^2)$$
$$(L_1-Q)+(1-\tau_0\,v_2^2)\,(\phi_2\,v_2-av_2/R)\;+\;\varepsilon\tau_1\,v^2\,(S_2+\tau_1^{-1})\}. \qquad \ldots (4.23)$$

Inverting the Laplace transform, we obtain

$$Z(R,\,\tau) = -\;\{P_1\,[1+\lambda_1'\,(\tau-R_1/v_1)]\,H(\tau-R_1/v_1)$$
$$+\,P_2\,[1+\lambda_2'\;(\tau-R_1/v_2)]\,H(\tau-R_1/v_2)\}/R(v_1^2-v_2^2) \quad \ldots (4.24)$$

$$U(R,\ \tau) = \{v_1^{-1} P_1 (1 - \tau_0 v_1^2) [(\tau - R_1/v_1) + (v_1/R + \lambda_1)(\tau - R_1/v_1)^2] \times H(\tau - R_1/v_1) + v_2^{-1} (1 - \tau_0 v_2^2) P_2 [(\tau - R_1/v_2) + (v_2/R + \lambda_2) \times (\tau - R_1/v_2)^2]\ H(\tau - R_1/v_2)\}/\varepsilon\, \tau_0 R(v_1^2 - v_2^2) \quad \text{... (4.25)}$$

$$\sigma_R (R, \tau) = \{\exp(-\phi_1 R_1) [(1 - \tau_0 v_2^2 + \varepsilon \tau_0 \eta v_2^2)(1 - \tau_0 v_1^2 - \varepsilon \tau_0 v_1^2) + \lambda_1^* (\tau - R_1/v_1)] H(\tau - R_1/v_1) + \exp(-\phi_2 R_1) \times [(1 - \tau_0 v_1^2 + \varepsilon \tau_0 \eta v_2^2)\ (1 - \tau_0 v_2^2 - \varepsilon \tau_0 v_2^2) + \lambda_2^* (\tau - R_1/v_2)] H(\tau - R_1/v_2)\}/R\varepsilon\, \tau_0 (v_1^2 - v_2^2) \quad \text{... (4.26)}$$

for L-S theory and

$$Z(R,\ \tau) = -\ \{P_1' [1 + (L_2 - Q - S_1)(\tau - R_1/v_1)] H(\tau - R_1/v_1) + P_2' [1 + (L_1 - Q - S_2)(\tau - R_1/v_2)] H(\tau - R_1/v_2)\}/R(v_1^2 - v_2^2) \quad \text{... (4.27)}$$

$$U(R,\ \tau) = \{v^{-1} P_1' (1 - \tau_0 v_1^2) [1 + (v_1/R + L_2 - Q)(\tau - R_1/v_1)] \times H(\tau - R_1/v_1) + v_2^{-1} P_2'\ (1 - \tau_0 v_2^2) [1 + v_2/R + L_1 - Q) \times (\tau - R_1/v_2)]\ H(\tau - R_1/v_2)\}/\varepsilon R\, (v_1^2 - v_2^2) \quad \text{... (4.28)}$$

$$\sigma_R (R, \tau) = \{\exp\ (-\ \phi_1 R_1) [1 - \tau_0 v_2^2 + \varepsilon \tau_1 \eta v_2^2)\ (1 - \tau_0 v_1^2 - \varepsilon \tau_1 v_1^2) \times \delta(\tau - R_1/v_1) + L_1^* H(\tau - R_1/v_1)] + \exp\ (-\phi_2 R_1) \times [(1 - \tau_0 v_1^2 + \varepsilon \tau_1 \eta v_1^2)\ (1 - \tau_0 v_2^2 - \varepsilon \tau_1 v_2^2)\ \delta(\tau - R_1/v_2) + L_2^* H(\tau - R_1/v_2)]\}/\varepsilon R\, (v_1^2 - v_2^2) \quad \text{... (4.29)}$$

for G-L theory, where

$$P_1 = v_1^2 (1 - \tau_0 v_2^2 + \varepsilon \eta \tau_0 v_2^2) \exp(-\phi_1 R_1),$$

$$P_2 = v_2^2 (1 - \tau_0 v_1^2 + \varepsilon \eta \tau_0 v_1^2) \exp(-\phi_2 R_1),$$

$$P_1' = v_1^2 (1 - \tau_0 v_2^2 + \varepsilon \eta \tau_1 v_2^2) \exp(-\phi_1 R_1),$$

$$P_2' = v_2^2 (1 - \tau_0 v_1^2 + \varepsilon \eta \tau_1 v_1^2) \exp(-\phi_2 R_1). \quad \text{... (4.30)}$$

5. Long Time Solutions

The long time solutions can be obtained by expanding the roots m_1^2 and m_2^2 of eqn. (3.7) for small values of p in the Taylor series. We obtain

$$m_1 = \sqrt{(1+\varepsilon)}\,\sqrt{p} + O(p^{3/2}), \qquad m_2 = (1+\varepsilon)^{-1/2}\,p + O(p^2).$$

Substituting these values of m_1 and m_2 into various relevant equations we can obtain $\overline{U}$ and $\overline{Z}$, which on inversion of Laplace transform provide us with the deformation and temperatrue. It is observed that m_1 and m_2 do not involve the thermal relaxation times τ_0 or τ_1, which ascertain that the "second sound" effects are short lived. Thus the small time solutions are of more physical importance than those of long time solutions.

6. Discussion of the Results at the Wave-Fronts

The short time solutions obtained above show that they consist of the two waves, dilatational and thermal travelling with velocity v_1 and v_2 respectively. The terms containing $H(\tau - R_1/v_1)$ represent the contribution of the elastic wave in the vicinity of its wave-front $R_1 = v_1\tau$, and terms with $H(\tau - R_1/v_2)$ represent contribution of the thermal wave in the vicinity of its wave front $R_1 = v_2\tau$. It is observed that the displacement is continuous on both the wave-fronts in the context of L-S theory in both the cases and for Case I in G-L theory but it is discontinuous for case-II in G-L theory. The temperature is found to be continuous on both the wave-fronts for Case I in G-L theory and is discontinuous elsewhere. The stress is discontinuous in each case. The discontinuities are given by

$$[Z^+ - Z^-]_{R_1 = v_1\tau} = \varepsilon\,\tau_0\,v_1^2\,v_2^2 \exp(-\phi_1 R_1)/R(v_1^2 - v_2^2)$$

$$[Z^+ - Z^-]_{R_1 = v_2\tau} = \varepsilon\,\tau_0\,v_1^2\,v_2^2 \exp(-\phi_2 R_1)/R(v_1^2 - v_2^2),$$

for Case I in L-S theory

$$[Z^+ - Z^-]_{R_1 = v_1\tau} = -P_1/R(v_1^2 - v_2^2)$$

$$[Z^+ - Z^-]_{R_1 = v_2\tau} = -P_2/R(v_1^2 - v_2^2),$$ for Case II in L-S theory

$$[Z^+ - Z^-]_{R_1 = v_1\tau} = -P_1'/R(v_1^2 - v_2^2)$$

$$[Z^+ - Z^-]_{R_1 = v_2\tau} = -P_2'/R(v_1^2 - v_2^2),$$ for Case II in G-L theory

$$[U^+ - U^-]_{R_1 = v_1\tau} = (1 - \tau_0 v_1^2)\,P_1'/\varepsilon R v_1 (v_1^2 - v_2^2)$$

$$[U^+ - U^-]_{R_1 = v_2\tau} = (1 - \tau_0 v_2^2)\,P_2'/\varepsilon R v_2 (v_1^2 - v_2^2),$$

for Case II in G-L theory

$$[\sigma_R^+ - \sigma_R^-]_{R_1 = v_1\tau} = -v_2^2(1 - \tau_0 v_1^2 + \varepsilon v_1^2 \tau_0)\exp(-\phi_1 R_1)/R(v_1^2 - v_2^2)$$

$$[\sigma_R^+ - \sigma_R^-]_{R_1 = v_2\tau} = -v_1^2(1 - \tau_0 v_2^2 + \varepsilon v_2^2 \tau_0)\exp(-\phi_2 R_1)/R(v_1^2 - v_2^2),$$

for Case I in L-S theory

$$[\sigma_R^+ - \sigma_R^-]_{R_1 = v_1\tau} = v_2^2(1 - \tau_0 v_1^2 + \varepsilon \tau_1 \eta v_1^2)\exp(-\phi_1 R_1)/R(v_1^2 - v_2^2)$$

$$[\sigma_R^+ - \sigma_R^-]_{R_1 = v_2\tau} = v_1^2(1 - \tau_0 v_2^2 + \varepsilon \tau_1 \eta v_2^2)\exp(-\phi_2 R_1)/R(v_1^2 - v_2^2)$$

for Case I in G-L theory

$$[\sigma_R^+ - \sigma_R^-]_{R_1 = v_1\tau} = (1 - \tau_0 v_2^2 + \varepsilon \tau_0 \eta v_2^2)(1 - \tau_0 v_1^2 - \varepsilon \tau_0 v_1^2) \times \exp(-\phi_1 R_1)/\varepsilon \tau_0 R(v_1^2 - v_2^2)$$

$$[\sigma_R^+ - \sigma_R^-]_{R_1 = v_2\tau} = (1 - \tau_0 v_1^2 + \varepsilon \tau_0 \eta v_1^2)(1 - \tau_0 v_2^2 - \varepsilon \tau_0 v_2^2) \times \exp(-\phi_2 R_1)/\varepsilon \tau_0 R(v_1^2 - v_2^2)$$

for Case II in L-S theory

$$[\sigma_R^+ - \sigma_R^-]_{R_1 = v_1\tau} = L_1^* \exp(-\phi_1 R_1)/\varepsilon \tau_0 R(v_1^2 - v_2^2)$$

$$[\sigma_R^+ - \sigma_R^-]_{R_1 = v_2\tau} = L_2^* \exp(-\phi_2 R_1)/\varepsilon \tau_0 R(v_1^2 - v_2^2)$$

for Case II in G-L theory.

These jumps in displacement, stress and temperature decay exponentially with time and also vanish when the radial distance R increases infinitely. In case of the conventional coupled theory of thermoelasticity i.e. for $\tau_1 = 0 = \tau_0$, we have

$$K_1 = 1 + \varepsilon,\ K_2 = 1,\ v_1 = 1,\ v_2 \to \infty,\ \phi_1 = \varepsilon/2,\ \phi_2 \to \infty.$$

The displacement and temperature are found to be continuous at both the wave-fronts for Case I and Case II. For $\tau_0 = 0$ and $\tau_1 \neq 0$, the displacement and temperature are discontinuous at the elastic wave-front in Case II, for G-L theory only. If we take $C_{11} = \lambda + 2\mu$, $C_{12} = \lambda = C_{13}$, $\alpha = \alpha_1 = \alpha_3$, $\beta_1 = \beta$, $K_1 = K$, then results reduce to those for isotropic material.

7. Numerical Results and Discussions

The theoretical results obtained in section 4 are also computed numerically for the zinc crystal for which the physical data is given by (Singh and Sharma[3])

$$\varepsilon = 0.221,\ C_{11} = 1.628 \times 10^{11} \times \text{Nm}^{-2},\ C_{12} = 0.362 \times 10^{11}\ \text{Nm}^{-2}$$

$C_{13} = 0.508 \times 10^{11}$ Nm^{-2}, $\beta = 5.75 \times 10^{6}$ Nm^{-2} deg^{-1},

$C_e = 3.9 \times 10^{2}$ J Kg^{-1} deg^{-1}

$K = K_1 = K_3 = 1.24 \times 10^{2}$ Wm^{-1} deg^{-1}, $\rho = 7.14 \times 10^{3}$ Kgm^{-3},

$T_0 = 296^{\circ}$ K, $\sigma_0 = 1.0 = \theta_0$

We take $\eta = 1$, $\tau_0 = 0, 0.02$; $\tau_1 = 0, 0.02, 0.05$ and compute displacement, stress and temperature for different values of the time τ and distance $R_1 = 1.0$.

The results so obtained have been represented graphically in Figs. 1-3. The magnitude of jumps in temperature at the elastic wave-front in case of normal load increases from negative values to become zero after finite values of time in L-S theory, where as these jumps in case of thermal shock decay exponentially tend to zero in both the theories at different values of relaxation times as revealed from Figs. 1(a) and 2(a) respectively. Figure 3(a) shows that the jumps in displacement at the elastic wave-front in case of thermal shocks at different values of thermal relaxation times in G-L theory increase exponentially to tend to zero at finite time and Fig. 3(b) reveals that these jumps decay exponentially indefinitely at the thermal wave front. Figures 1(b) and 2(b) represents that the magnitudes of jumps in temperature at the thermal wave front decay experientially and may assume indefinite negative value with the passage of time in L-S theory for normal load and in both the theories for thermal shock respectively.

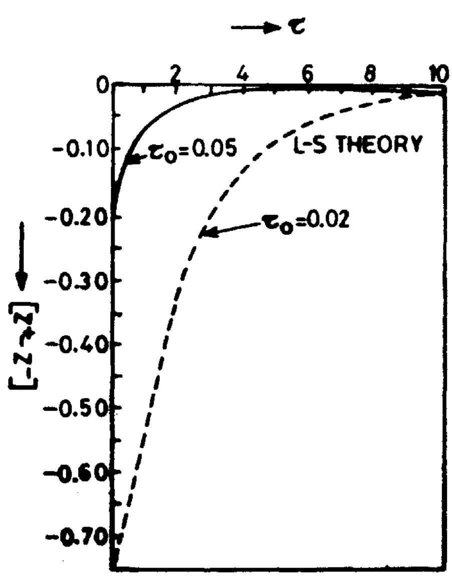

FIG. 1(a). Variation of jumps in temperature at elastic wave front (normal load).

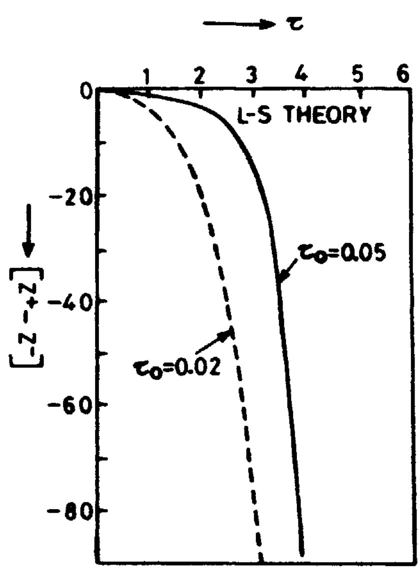

FIG. 1(b). Variation of jumps in temperature at thermal wave front (normal load).

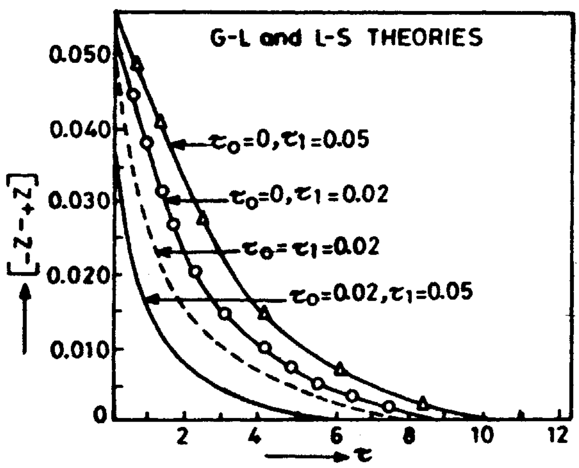

FIG. 2(a). Variation of jumps in temperature at elastic wave front (thermal shock).

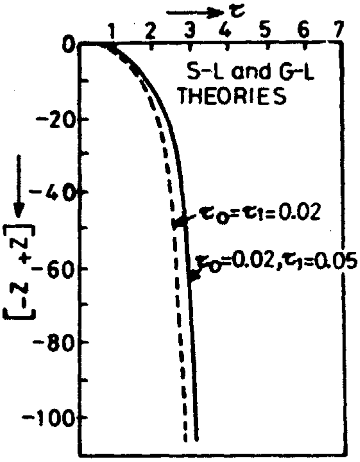

FIG. 2(b). Variation of jumps in temperature at thermal wave front (thermal shock).

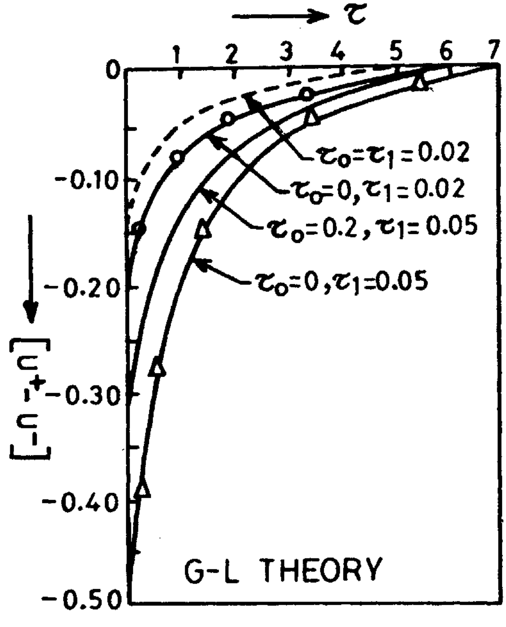

FIG. 3(a). Variation of jumps in displacement at elastic wave front (thermal shock).

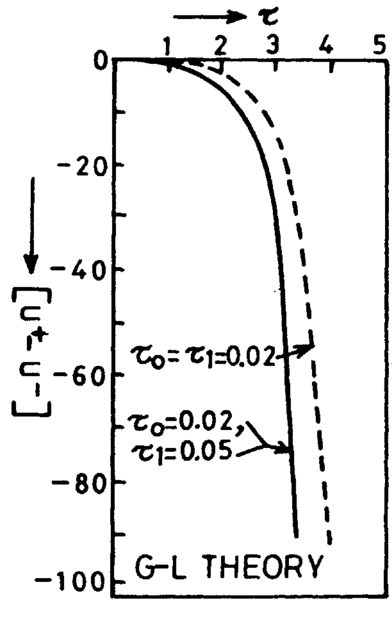

FIG. 3(b). Variation of jumps in displacement at thermal wave front (thermal shock).

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