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MULTIOBJECTIVE FRACTIONAL PROGRAMMING UNDER GENERALIZED INVEXITY

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We obtain some optimality conditions and duality results concerned with multiobjective fractional nonlinear programming problem under generalized invexity assumptions on the objective and constraint functions. In particular we deal with (ρ, b)-invex functions and their generalizations.

1. INTRODUCTION

A great deal of work has been done in multiobjective programming problem involving non convex functions. Duality results, necessary and sufficient conditions of Kuhn-Tucker and Fritz-John type have been obtained for multiobjective programming problem involving non convex functions by many researchers. Among many such contributions, works of Weir[15, 16], Egudo and Hanson[3], Weir and Mond[17], Kual *et al.*[8], Suneja and Gupta[12] and Suneja *et al.*[11] fall in this category.

Various generalizations of convex functions have appeared in the literature. Vial[14] generalized the concept of convex function by introducing ρ-convex functions. The generalization of ρ-convex functions to ρ-pseudoconvex and ρ-quasiconvex functions was given by Jeyakumar[6]. A class of functions called b-vex functions were introduced by Bector and Singh[1]. In recent times some authors have defined more general class of functions labelled as (ρ, b)-invex functions. This class is defined by relaxing the definitions of ρ-invex and b-vex functions. Namely certain examples for a wide variety of functions satisfying b-vexity functions can be seen from Bector *et al.*[2] and Suneja and Lalitha[13]. For ρ-invex functions many references are available in the current literature.

In this paper we consider (ρ, b)-invexity and its generalizations and present some optimality conditions for a multiobjective non linear fractional programming problem. The duality results are also obtained relative to the Schaible type dual[10] of the primal problem. The duality results are stated by assuming the functions involved are (ρ, b)-invex, (ρ-b)-psuedoinvex and (ρ, b)-quasiinvex. In fact Preda[9] has used (ρ, b)-vex functions to develop duality results for optimization problems for n-set functions.

The multiobjective nonlinear fractional programming problem we consider in this paper is the following :

$$\text{(FP)} \qquad \text{Minimize} \left\{ Q(x) = \frac{f(x)}{g(x)}\,; \quad x \in X \right\}$$

$$\text{subject to } h_j(x) \leq 0;\ j = 1, 2, \ldots, m;\ x \in S$$

where

$$\frac{f(x)}{g(x)} = \left\{ \frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_k(x)}{g_k(x)} \right\}^t$$

(the symbol t indicates the transpose operation), the feasible set $X = \{x \in S;\ h_j(x) \leq 0;\ j = 1, 2, \ldots, m\}$ and $S \subseteq \mathbb{R}^n$.

Here $f_i : S \to \mathbb{R}$, $g_i : S \to \mathbb{R}$; $i = 1, 2, \ldots, k$ and $h_j : S \to \mathbb{R}$, $j = 1, 2, \ldots, m$ are all assumed to be differentiable. It is further assumed that

(i) $g_i(x) > 0$ for all $x \in S$ and each $i = 1, 2, \ldots, k$;

(ii) $X \neq \phi$.

In this paper, the concept of proper efficiency and the parametric approach are used to obtain the optimality conditions and duality results. This paper is organised as follows. Section 2 gives definitions of (ρ, b)-invexity and its generalizations. In section 3 we present the necessary and sufficient optimality conditions. We formulate the dual problem of Schaible type for multiobjective programming and present a number of duality results in section 4.

2. Definitions and Preliminaries

For $x, y \in \mathbb{R}^n$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \ldots, n$; by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$ and $x_j < y_j$ for at least one j, $1 \leq j \leq n$ and for $x, y \in \mathbb{R}$, $x < y$ is used in the usual sense.

We shall now define the notion of (ρ, b)-invexity. Let $K = \{1, 2, \ldots, k\}$ and $M = \{1, 2, \ldots, m\}$.

Definition 2.1 — Let f be a numerical function defined on an open set $S \subseteq \mathbb{R}^n$ and let f be differentiable at $\bar{x} \in S$. Let ρ be a real number and $\eta, \theta : S \times S \to \mathbb{R}^n$ and $b : X \times S \to \mathbb{R}_+$ be functions. Then the function f is said to be

(i) (ρ, b)-invex at $\bar{x}$ with respect to S, η, θ and b or briefly, (ρ, b)-invex at $\bar{x}$ if for each $x \in S$

$$b(x, \bar{x})\ [f(x) - f(\bar{x})] \geq \eta^t (x, \bar{x})\ \nabla f(\bar{x}) + \rho \, \| \theta (x, \bar{x}) \|^2 \qquad \ldots (2.1)$$

(ii) (ρ, b)-pseudoinvex at $\bar{x}$ if for each $x \in S$

$$b(x, \bar{x})\ \eta^t (x, \bar{x})\ \nabla f(\bar{x}) + \rho \, \| \theta (x, \bar{x}) \|^2 \geq 0 \qquad \ldots (2.2)$$

implies

$$f(x) \geq f(\bar{x}) \qquad \ldots (2.3)$$

(iii) (ρ, b)-quasiinvex at $\bar{x}$ if for each $x \in S$

$$f(x) \leq f(\bar{x})$$

implies

$$b(x, \bar{x})\ \eta'(x, \bar{x})\ \nabla f(\bar{x}) + \rho \,\|\theta(x, \bar{x})\|^2 \leq 0.$$

Remark 2.2 : f is said to be

(i) (ρ, b)-strictly invex at $\bar{x}$ if strict inequality holds in (2.1).

(ii) (ρ, b)-strictly pseudoinvex at $\bar{x}$ if for each $x \in S$, $x \neq \bar{x}$, strict inequality holds in (2.3).

Definition 2.3 — A point $\bar{x} \in X$ is said to be efficient solution for (FP) if there does not exist any $x \in X$ such that $Q(x) \leq Q(\bar{x})$.

Definition 2.4 — A point $\bar{x} \in X$ is said to be properly efficient for (FP) if it is efficient for (FP) and if there exists a scalar $N > 0$ such that for each i

$$Q_i(\bar{x}) - Q_i(x) \leq N\,[Q_j(x) - Q_j(\bar{x})]$$

for some j such that $Q_j(x) > Q_j(\bar{x})$ whenever x is feasible for (FP) and $Q_i(x) < Q_i(\bar{x})$.

3. Optimal Conditions

Following Kaul and Lyall[7], Suneja and Gupta[12] we consider the following parametric multiobjective optimization problem $(\mathrm{FP})_v$ for each $v \in \mathbb{R}^k_+$ where $\mathbb{R}^k_+$ denotes the positive orthant of $\mathbb{R}^k$.

$(\mathrm{FP})_v$ Minimize $[f_1(x) - v_1\, g_1(x), \ldots, f_k(x) - v_k\, g_k(x)]$

subject to $h_j(x) \leq 0,\ j \in M : x \in S.$

We now have the following lemma connecting the properly efficient solutions of (FP) and $(\mathrm{FP})_v$ which can be proved on the same lines as Lemma 1 of Kaul and Lyall[7].

Lemma 3.1 — Let $\bar{x}$ be properly efficient for (FP). Then there exists $\bar{v} \in \mathbb{R}^k_+$ such that $\bar{x}$ is properly efficient for $(\mathrm{FP})_{\bar{v}}$. Conversely, if $\bar{x}$ is properly efficient for $(\mathrm{FP})_{\bar{v}}$ where $\bar{v}_i = Q_i(\bar{x})$, $i \in K$ then $\bar{x}$ is properly efficient for (FP).

Following Geoffrion[4] we now consider the scalar programme corresponding to $(\mathrm{FP})_{\bar{v}}$ given by

$(\mathrm{FP})^{\lambda}_{\bar{v}}$ Minimize $\displaystyle\sum_{i=1}^{k} \lambda_i\, [f_i(x) - \bar{v}_i\, g_i(x)]$

subject to $h_j(x) \leq 0,\ j \in M,\ x \in S.$

We have the following result on the lines of Geoffrion[4].

Lemma 3.2 — If $\bar{x}$ is an optimal solution for $(FP)_{\bar{v}}^{\lambda}$ for some $\lambda \in \mathbb{R}^k$ with strictly positive components then $\bar{x}$ is properly efficient for (FP).

Under the assumption of (ρ, b)-invex, (ρ, b)-quasiinvex and (ρ, b)-pseudoinvex objective and constraint functions we now give a number of sufficient optimality criteria for a feasible $\bar{x}$ for (FP) to be properly efficient for (FP). These generalize the sufficient optimality criteria given by Suneja *et al.*[11].

Theorem 3.3 — Let $\bar{x}$ be a feasible solution for (FP). Assume that there exists $\bar{v}$ (as given by Lemma 3.1) , $\bar{\lambda} \in \mathbb{R}_+^k$ and $\bar{\mu}_j \geq 0$ for each $j \in I = \{j : h_j(\bar{x}) = 0\}$ the set of indices of active constraint functions at $\bar{x}$, such that

$$\sum_{i=1}^{k} \bar{\lambda}_i [\nabla f_i(\bar{x}) - \bar{v}_i \nabla g_i(\bar{x})] + \sum_{j \in I} \bar{\mu}_j \nabla h_j(\bar{x}) = 0. \qquad \ldots (3.1)$$

Then $\bar{x}$ is properly efficient for (FP) if any of the following holds at $\bar{x}$ with respect to same η, θ and b :

(a) $f_i, -g_i$ for each $i \in K$ and h_j for each $j \in I$ are (ρ_i, b)-invex, (σ_i, b)-invex and (γ_j, b_j)-invex respectively such that

$$\sum_{i=1}^{k} \bar{\lambda}_i [\rho_i + \bar{v}_i \sigma_i] + \sum_{j \in I} \bar{\mu}_j \gamma_j \geq 0 \qquad \ldots (3.2)$$

and for all $x \in X$, $b(x, \bar{x}) > 0$

(b) $\sum_{i=1}^{k} \bar{\lambda}_i (f_i - \bar{v}_i g_i)$ is (ρ, b)-pseudoinvex and h_j for each $j \in I$ is (γ_j, b)-quasiinvex such that

$$\sum_{j \in I} \bar{\mu}_j \gamma_j + \rho \geq 0 \qquad \ldots (3.3)$$

(c) $\sum_{i=1}^{k} \bar{\lambda}_i (f_i - \bar{v}_i g_i)$ is (ρ, b)-pseudoinvex and $\sum_{j \in I} \bar{\mu}_j h_j$ is (γ, b)-quasiinvex such that

$$\rho + \gamma \geq 0. \qquad \ldots (3.4)$$

PROOF : (a) For any feasible x of (FP) and $j \in I$

$$h_j(x) \leq 0 = h_j(\bar{x}).$$

Since h_j, $j \in I$ is (γ_j, b_j)-invex with respect to η, θ at $\bar{x}$, it follows from above that for $i \in I$

$$0 \geq b_j(x, \bar{x}) [h_j(x) - h_j(\bar{x})] \geq \eta^t (x, \bar{x}) \ \nabla h_j(\bar{x}) + \gamma_j \| \theta(x, \bar{x}) \|^2.$$

As $\bar{\mu}_j \geq 0$ for $j \in I$ we have

$$\eta^t(x,\bar{x}) \sum_{j\in I} \bar{\mu}_j \nabla h_j(\bar{x}) + \sum_{j\in I} \bar{\mu}_j \gamma_j \|\theta(x,\bar{x})\|^2 \leq 0.$$

On using (3.1) and (3.2), we obtain

$$\eta^t(x,\bar{x}) \sum_{i=1}^{k} \bar{\lambda}_i [\nabla f_i(\bar{x}) - \bar{v}_i \nabla g_i(\bar{x})] + \sum_{i=1}^{k} \bar{\lambda}_i [\rho_i + \bar{v}_i \sigma_i] \|\theta(x,\bar{x})\|^2 \geq 0.$$

Now the (ρ_i, b)-invexity of f_i and the (σ_i, b)-invexity of $-g_i$ give us

$$\sum_{i=1}^{k} \bar{\lambda}_i b(x,\bar{x}) [f_i(x) - \bar{v}_i g_i(x)] - \sum_{i=1}^{k} \bar{\lambda}_i b(x,\bar{x}) [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] \geq 0.$$

Since $b(x, \bar{x}) > 0$ we get

$$\sum_{i=1}^{k} \bar{\lambda}_i [f_i(x) - \bar{v}_i g_i(x)] \geq \sum_{i=1}^{k} \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})].$$

This shows that $\bar{x}$ is optimal for $(\mathrm{FP})^{\bar{\lambda}}_{\bar{v}}$. Since $\bar{\lambda}_i > 0$, Lemma 3.2 shows that $\bar{x}$ is properly efficient for (FP).

(b) For any feasible x of (FP), $h_j(x) \leq 0 = h_j(\bar{x})$ for all $j \in I$. Hence (γ_j, b)-quasiinvexity of h_j implies

$$b(x, \bar{x}) \; \nabla h_j(\bar{x}) \eta^t(x,\bar{x}) + \gamma_j \|\theta(x,\bar{x})\|^2 \leq 0$$

i.e.
$$b(x,\bar{x}) \eta^t(x,\bar{x}) \sum_{j\in I} \bar{\mu}_j \nabla h_j(\bar{x}) + \sum_{j\in I} \bar{\mu}_j \gamma_j \|\theta(x,\bar{x})\|^2 \leq 0.$$

On using (3.1) and (3.3),

$$b(x,\bar{x}) \sum_{i\in 1}^{k} \bar{\lambda}_i (\nabla f_i(\bar{x}) - \bar{v}_i \nabla g_i(\bar{x})] \eta^t(x,\bar{x}) + \rho \|\theta(x,\bar{x})\|^2 \geq 0. \quad \dots (3.5)$$

Now (ρ, b)-pseudo invexity of $\sum_{i=1}^{k} \bar{\lambda}_i (f_i - \bar{v}_i g_i)$ shows that $\bar{x}$ is optimal for $(\mathrm{FP})^{\bar{\lambda}}_{\bar{v}}$. Hence by Lemma 3.2, $\bar{x}$ is properly efficient for (FP).

(c) As in (b), (γ, b)-quasi invexity of $\sum_{j \in I} \bar{\mu}_j h_j$ implies

$$b(x, \bar{x}) \sum_{j \in I} \bar{\mu}_j \nabla h_j(\bar{x}) \eta^t(x, \bar{x}) + \gamma \| \theta(x, \bar{x}) \|^2 \leq 0.$$

On using (3.1) and (3.4) we get (3.5) and the rest of the proof follows on the same lines as that of (b).

Theorem 3.4 — Let $\bar{x}$ be properly efficient for (FP). Then there exist $\bar{\lambda}, \bar{\mu}$ and $\bar{v}$ (as are given by Theorem 3.3) such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, v)$ satisfies (3.1) if

(*) $\exists$ some $\bar{x} \in S$ such that $h_j(\bar{x}) < 0$ for $j \in M$ and if either of the following holds on S with respect to same η, θ and b :

(a) h_j is (γ_j, b)-invex for each $j \in I$ with $\gamma_j \geq 0$, and $b(x, x) > 0$.

(b) h_j is (γ_j, b)-pseudoinvex for each $j \in I$ with $\gamma_j \geq 0$, and $b(x, \bar{x}) > 0$.

PROOF : The proof goes on the same lines as that of Theorem 4 and Theorem 5 of Suneja *et al.*[11].

4. Duality Theorems

Weir[16] gave the multiobjective analogue of Jagannathan[5] and Schaible[10] dual for (FP) and proved the duality results under the conditions of convexity. Then after, Kaul and Lyall[7] have considered the Schaible[10] type dual and proved duality results under the assumptions of strict pseudo convexity. In a similar way Suneja and Gupta[12] proved duality results under the assumptions of η-pseudoconvexity, η-strict pseudo convexity and η-convexity. We associate the following Schaible type[10] dual multiple objective maximization problem given by Kaul and Lyall[7] to (FP).

(FD) Max $v = (v_1, v_2, \ldots, v_k)$

$$\text{subject to } \sum_{i=1}^{k} \lambda_i [\nabla f_i(u) - v_i \nabla g_i(u)] + \sum_{j=1}^{k} \mu_j \nabla h_j(u) = 0 \quad \ldots (4.1)$$

$$\sum_{i=1}^{k} \lambda_i [f_i(u) - v_i g_i(u)] \geq 0 \quad \ldots (4.2)$$

$$\sum_{j=1}^{m} \mu_j h_j(u) \geq 0 \quad \ldots (4.3)$$

where $u \in S$, $v, \lambda \in \mathbb{R}_+^k$ and $\mu \in \mathbb{R}_+^m$.

Now we give various duality theorems between programmes (FP) and (FD) under the assumptions of (ρ, b)-invexity and its generalizations on the objective and constraint functions.

Theorem 4.1 (Weak Duality) — Assume that for all feasible x for (FP) and for all feasible (u, λ, μ, v) for (FD) either of the following holds at u on X with respect to the same η, θ and b :

(i) $f_i, -g_i$ for each $i \in K$ and h_j for each $j \in M$ are (σ_i, b)-invex (ρ_i, b)-invex and (γ_j, b)-invex respectively such that

$$\sum_{i=1}^{k} \lambda_i (\rho_i + v_i \sigma_i) + \sum_{j=1}^{m} \mu_j \gamma_j \geq 0 \qquad \text{... (4.4)}$$

and for all $x \in X$, $b(x, \bar{x}) > 0$.

(ii) $\sum_{i=1}^{k} \lambda_i (f_i - v_i g_i)$ is (ρ, b)-pseudo invex and $\sum_{j=1}^{m} \mu_j h_j$ is (γ, b)-quasiinvex such that

$$\rho + \gamma \geq 0. \qquad \text{... (4.5)}$$

Then $\quad Q(x) \leq v$.

PROOF : (i) From the respective definitions of (ρ_i, b)-invexity (σ_i, b)-invexity and (γ_j, b)-invexity of $f_i, -g_i$ and h_j, we get

$$b(x, u) \sum_{i=1}^{k} \lambda_i [f_i(x) - v_i g_i(x)] \geq b(x, u) \sum_{i=1}^{k} \lambda_i [f_i(u) - v_i g_i(u)]$$

$$+ \sum_{i=1}^{k} \lambda_i [\nabla f_i(u) - v_i \nabla g_i(u)] \eta^t(x, u) + \sum_{i=1}^{k} \lambda_i \| \theta(x, u) \|^2 (\rho_i + v_i \sigma_i)$$

$$\geq - \sum_{j=1}^{m} \mu_j [\nabla h_j(u) \eta^t(x, u) + \gamma_j \| \theta(x, u) \|^2]$$

(on using the duality constraints (4.1) and (4.2), (4.4) and $b(x, u) > 0$).

$$\geq - \sum_{j=1}^{m} \mu_j b(x, u) [h_j(x) - h_j(u)] \qquad \text{(from } (\gamma_j, b)\text{-invexity of } h_j)$$

since $b(x, u) > 0$, on applying (4.3) and the duality constraint of (FP) we obtain

$$\sum_{i=1}^{k} \lambda_i [f_i(x) - v_i g_i(x)] \geq 0. \qquad \text{... (4.6)}$$

Now suppose contrary to the result of the theorem hold i.e., $Q(x) \leq v$. Then there exist some $j \in K$ such that $Q_j(x) < v_j$ and $Q_i(x) \leq v_i$ for all $i \in K$. Which gives $f_j(x) - v_j g_j(x) < 0$ for some $j \in K$ and $f_i(x) - v_i g_i(x) \leq 0$ for all $i \in K$.

Since $\quad \lambda_i > 0, \sum_{i=1}^{k} \lambda_i [f_i(x) - v_i g_i(x)] < 0$

which contradicts (4.6).

(ii) Since x is feasible for (FP), u is feasible to (FD) and $\mu_j \geq 0$ for all $j \in M$, we have

$$\sum_{j=1}^{m} \mu_j h_j(x) \leq \sum_{j=1}^{m} \mu_j h_j(u).$$

The (γ, b) quasiinvexity of $\sum_{j=1}^{m} \mu_j h_j$ gives

$$b(x, u) \sum_{j=1}^{m} \mu_j \nabla h_j(u) \eta'(x, u) + \gamma \| \theta(x, u) \|^2 \leq 0.$$

On using the duality constraint (4.1) and (ρ, b)-pseudoinvexity

$$\sum_{i=1}^{k} \lambda_i (f_i - v_i g_i)(x) \geq \sum_{i=1}^{k} \lambda_i (f_i - v_i g_i)(u)$$

i.e., $$\sum_{i=1}^{k} \lambda_i (f_i - v_i g_i)(x) \geq 0 \quad \text{from (4.2).}$$

The rest of the proof follows on the same lines as that of (i).

Theorem 4.2 (Strong Duality) — Assume (*) of Theorem 3.4 and that either of the following holds on S with respect to same η, θ and b :

(a) (i) of Theorem (4.1) with $\gamma_j \geq 0$ for each $j \in I$;

(b) $\sum_{i=1}^{k} \lambda_i (f_i - v_i g_i)$ is (ρ, b)-pseudoinvex and $\sum_{j=1}^{m} \mu_j h_j$ is (σ, b)-quasiinvex such that $\rho + \sigma \geq 0$ and for each $j \in I$, h_j is (γ_j, b)-quasiinvex with $\gamma_j \geq 0$.

Then if $\bar{x}$ is properly efficient for (FD) then there exist $\bar{\lambda}, \bar{\mu}$ and $\bar{v}$ (as are given by Theorem 3.1) such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is properly efficient for (FD) and the objective values of (FP) and (FD) are equal.

PROOF : From Theorem 3.4, there exist $\bar{\lambda}_i > 0$, $i \in K$, $\bar{\mu}_j \geq 0$ $j \in I$, $\bar{v}_i \geq 0$, $i \in K$ such that (3.1) is satisfied (where $\bar{v}_i$ is given as in Lemma 3.1) setting $\bar{\mu}_j = 0$ for $j \notin I$ gives that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is feasible for (FD).

Suppose that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is not efficient for (FD) then there exists (u, λ, μ, v) feasible for (FD) and an index j such that $\bar{v}_j < v_j$ and $\bar{v}_i \leq v_i$ for all $i \neq j$ for $i \in K$. Since $Q_i(\bar{x}) = \bar{v}_i$ we get a contradiction to weak duality Theorem 4.1.

Now suppose that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is not properly efficient for (FD). By following exactly on the same lines as that of Theorem 7 of Edugo and Hanson[3] we get again a contradiction to the weak duality Theorem 4.1 Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is properly efficient for (FD).

The equality of objective functions (FP) and (FD) follows from $Q_i(\bar{x}) = \bar{v}_i$ for $i \in K$.

(b) The proof follows on the same lines as that of (a) of the theorem.

Theorem 4.3 (Converse Duality) — Let $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{v})$ be feasible for (FD) and that there exists a feasible solution $\bar{x}$ for (FP) such that

$$\sum_{i=1}^{k} \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] = 0. \qquad \text{... (4.7)}$$

Then $\bar{x}$ is properly efficient for (FP) if either of the following holds at $\bar{u}$ with respect to X and the same η, θ and b :

(a) $f_i, -g_i$ for $i \in K$ and h_j for each $j \in M$ are (ρ_i, b)-invex (σ_i, b)-invex and (γ_j, b)-invex respectively such that

$$\sum_{i=1}^{k} \bar{\lambda}_i (\rho_i + \bar{v}_i \sigma_i) + \sum_{i=1}^{k} \bar{\mu}_j \gamma_j \geq 0$$

(b) (ii) of Theorem 4.1.

PROOF : (a) By following on the similar lines as that of Theorem 4.1 we get

$$\sum_{i=1}^{k} \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] \geq 0.$$

On using (4.7) and Lemma 3.2 we conclude that $\bar{x}$ is properly efficient for (FP).

(b) The proof is similar as that of (a) of the theorem.

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